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Hidden Grassmann structure in the XXZ model III: introducing the Matsubara direction

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Abstract
We address the problem of computing temperature correlation functions of the XXZ chain, within the approach developed in our previous works. In this paper we calculate the expected values of a fermionic basis of quasi-local operators, in the infinite volume limit while keeping the Matsubara (or Trotter) direction finite. The result is expressed in terms of two basic quantities: a ratio $\rho(\zeta)$ of transfer matrix eigenvalues and a nearest neighbour correlator $\omega(\zeta, \xi)$. We explain that the latter is interpreted as the canonical second kind differential in the theory of deformed Abelian integrals.

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1. Introduction

The present paper is a continuation of the paper [2], which was written almost a year ago and was dedicated to the memory of Alyosha Zamolodchikov. It so happens that the topic we discuss this time is not too far from a domain in which he made giant footsteps. So, life goes on, but there stays a painful sorrow caused by his early death.

Consider the XXZ spin chain with the Hamiltonian

$$H = \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(\sigma_k^1\sigma_{k+1}^1 + \sigma_k^2\sigma_{k+1}^2 + \Delta\sigma_k^3\sigma_{k+1}^3\right), \qquad \Delta = \frac{1}{2}(q+q^{-1}), \tag{1.1}$$

where σ^a ($a = 1, 2, 3$) are the Pauli matrices. To avoid technicalities, in this Introduction let us accept (1.1) as a formal object acting on $\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty}\mathbb{C}^2$. We shall touch upon the

[5] Membre du CNRS.

limit from a finite chain in the body of the text. In the papers [1, 2], we studied the vacuum expectation values (VEVs)

$$\langle q^{2\alpha S(0)}\mathcal{O}\rangle_{XXZ} = \frac{\langle \mathrm{vac}|q^{2\alpha S(0)}\mathcal{O}|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{2\alpha S(0)}|\mathrm{vac}\rangle}. \tag{1.2}$$

Here $|\mathrm{vac}\rangle$ denotes the ground-state eigenvector, $S(k) = \frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^3$, and $\mathcal{O}$ is a local operator. We have obtained a description of (1.2) in terms of fermionic operators. For that purpose, it was essential to consider operators of the form $q^{2\alpha S(0)}\mathcal{O}$, which we call quasi-local operators with tail α.

An important generalization of our results was proposed by Boos, Göhmann, Klümper and Suzuki [4]. They gave evidences that our fermionic description works equally well in the presence of a finite temperature and a non-zero magnetic field:

$$\langle q^{2\alpha S(0)}\mathcal{O}\rangle_{XXZ,\beta,h} = \frac{\mathrm{Tr}_{\mathrm{S}}(\mathrm{e}^{-\beta H+hS}q^{2\alpha S(0)}\mathcal{O})}{\mathrm{Tr}_{\mathrm{S}}(\mathrm{e}^{-\beta H+hS}q^{2\alpha S(0)})}, \tag{1.3}$$

where Tr_{S} stands for the trace on $\mathfrak{H}_{\mathrm{S}}$. For $\beta\to\infty$ and $h=0$, the expectation value (1.3) reduces to (1.2). For us this was quite an exciting development, because it shows that the fermionic structure is not a peculiarity of VEVs, but is rather a reflection of a symmetry hidden deep in the model. It should be said that in the paper [4] the expectation values (1.3) were not considered in full generality. The formula expressing them in terms of fermionic operators was formulated as a conjecture, which was checked in some particular cases but was left unproved.

The first question which we asked ourselves was, why not to add other local integrals of motion to $-\beta H + hS$ in (1.3). The physical meaning of such a generalization is obscure, but it should be possible for integrable models. This question, together with an intuition coming from the papers [6, 9], led to the following generalization of (1.3). Along with the space $\mathfrak{H}_{\mathrm{S}}$, consider the Matsubara space $\mathfrak{H}_{\mathbf{M}}$,

$$\mathfrak{H}_{\mathbf{M}} = \mathbb{C}^{2s_1+1}\otimes\cdots\otimes\mathbb{C}^{2s_{\mathbf{n}}+1}, \tag{1.4}$$

with an arbitrary spin $s_{\mathbf{m}}$ and a spectral parameter $\tau_{\mathbf{m}}$ attached to each component. The generalization of (1.3) is given by the following linear functional:

$$Z^{\kappa}\{q^{2\alpha S(0)}\mathcal{O}\} = \frac{\mathrm{Tr}_{\mathrm{S}}\,\mathrm{Tr}_{\mathbf{M}}(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O})}{\mathrm{Tr}_{\mathrm{S}}\,\mathrm{Tr}_{\mathbf{M}}(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)})}. \tag{1.5}$$

Here $T_{S,\mathbf{M}}$ denotes the monodromy matrix associated with $\mathfrak{H}_{\mathrm{S}}\otimes\mathfrak{H}_{\mathbf{M}}$ (see (2.2)).

The idea behind the generalization (1.5) is simple: for whichever spins and $\tau_{\mathbf{m}}$ that we put in the Matusbara direction, $\mathrm{Tr}_{\mathbf{M}}(T_{\mathrm{S},\mathbf{M}})$ commutes with H_{XXZ}. One expects that using cleverly this arbitrariness in the definition of $\mathfrak{H}_{\mathbf{M}}$, it should be possible to reproduce any function of local integrals of motion under the trace. In particular, in order to reproduce (1.3) from (1.5), one has to take special inhomogeneities and then to consider the limit $\mathbf{n}\to\infty$. This point is explained in detail in [6, 9].[6] In the present paper, we compute Z^{κ} for finite $\mathbf{n}$, leaving the discussion of the limit for future publication. We would like to emphasise, however, that this limit is not complicated. For finite $\mathbf{n}$, Z^{κ} will be expressed in terms of only two functions, $\rho(\zeta)$, $\omega(\zeta,\xi)$ (see (1.12) below) and one needs only to take the limit of them. Let us explain all that in some more details, starting from our fermionic operators.

For the moment we forget about the Matsubara direction, and concentrate on the description of the operators acting on $\mathfrak{H}_{\mathrm{S}}$. The logic of our papers [1, 2] is close to that of CFT: we describe the space of quasi-local operators as a module created from the primary

[6] With an appropriate choice of $\tau_{\mathbf{m}}$, (1.5) reproduces the trace of a 'quantum' transfer matrix of [6, 9].

field $q^{2\alpha S(0)}$ by creation operators. We recall below the main features of the construction in [2].

We say that $X = q^{2\alpha S(0)}\mathcal{O}$ is a quasi-local operator with tail α if it stabilizes outside some finite interval of the infinite chain: to $q^{\alpha\sigma_j^3}$ on the left and to I_j on the right. The minimal interval with this property is called the support of X. The spin of X is the eigenvalue of $\mathbb{S}(\cdot) = [S, \cdot]$ where $S = S(\infty)$ is the total spin operator. We denote by $\mathcal{W}_\alpha$ the space of quasi-local operators with tail α, and by $\mathcal{W}_{\alpha,s}$ its subspace of operators of spin $s \in \mathbb{Z}$. Consider the space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}.$$

On this space we defined the creation operators $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ and annihilation operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$. These are one-parameter families of operators of the form

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{t}_p^*,$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha+2}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{b}_p^*, \qquad \mathbf{c}^*(\zeta) = \zeta^{-\alpha-2}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{c}_p^*,$$

$$\mathbf{b}(\zeta) = \zeta^{-\alpha}\sum_{p=0}^{\infty}(\zeta^2-1)^{-p}\mathbf{b}_p, \qquad \mathbf{c}(\zeta) = \zeta^{\alpha}\sum_{p=0}^{\infty}(\zeta^2-1)^{-p}\mathbf{c}_p.$$

The operator $\mathbf{t}^*(\zeta)$ is in the centre of our algebra of creation–annihilation operators,

$$[\mathbf{t}^*(\zeta_1), \mathbf{t}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{c}^*(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}^*(\zeta_2)] = 0,$$
$$[\mathbf{t}^*(\zeta_1), \mathbf{c}(\zeta_2)] = [\mathbf{t}^*(\zeta_1), \mathbf{b}(\zeta_2)] = 0.$$

The rest of the operators $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}^*$, $\mathbf{c}^*$ are fermionic. The only non-vanishing anti-commutators are

$$[\mathbf{b}(\zeta_1), \mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1, \alpha), \qquad [\mathbf{c}(\zeta_1), \mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2, \alpha),$$

where

$$\psi(\zeta, \alpha) = \zeta^\alpha \frac{\zeta^2+1}{2(\zeta^2-1)}. \tag{1.6}$$

Each Fourier mode has the block structure

$$\begin{aligned} &\mathbf{t}_p^* : \mathcal{W}_{\alpha-s,s} \to \mathcal{W}_{\alpha-s,s} \\ &\mathbf{b}_p^*, \mathbf{c}_p : \mathcal{W}_{\alpha-s+1,s-1} \to \mathcal{W}_{\alpha-s,s}, \qquad \mathbf{c}_p^*, \mathbf{b}_p : \mathcal{W}_{\alpha-s-1,s+1} \to \mathcal{W}_{\alpha-s,s}. \end{aligned} \tag{1.7}$$

Among them, $\boldsymbol{\tau} = \mathbf{t}_1^*/2$ plays a special role. It is the right shift by one site along the chain. Consider the set of operators

$$\boldsymbol{\tau}^m \mathbf{t}_{p_1}^* \cdots \mathbf{t}_{p_j}^* \mathbf{b}_{q_1}^* \cdots \mathbf{b}_{q_k}^* \mathbf{c}_{r_1}^* \cdots \mathbf{c}_{r_k}^* (q^{2\alpha S(0)}), \tag{1.8}$$

where $m \in \mathbb{Z}$, $j, k \in \mathbb{Z}_{\geqslant 0}$, $p_1 \geqslant \cdots \geqslant p_j \geqslant 2$, $q_1 > \cdots > q_k \geqslant 1$ and $r_1 > \cdots > r_k \geqslant 1$. It can be shown that (1.8) constitutes a basis of $\mathcal{W}_{\alpha,0}$ (we postpone the proof to other publication).

Now we start to consider the spaces $\mathfrak{H}_\mathbf{S}$ and $\mathfrak{H}_\mathbf{M}$ together. We shall prove that

$$Z^\kappa\{\mathbf{t}^*(\zeta)(X)\} = 2\rho(\zeta)Z^\kappa\{X\}, \tag{1.9}$$

$$Z^\kappa\{\mathbf{b}^*(\zeta)(X)\} = \frac{1}{2\pi i}\oint_\Gamma \omega(\zeta, \xi) Z^\kappa\{\mathbf{c}(\xi)(X)\}\frac{d\xi^2}{\xi^2}, \tag{1.10}$$

$$Z^{\kappa}\{\mathbf{c}^{*}(\zeta)(X)\} = -\frac{1}{2\pi \mathrm{i}}\oint_{\Gamma}\omega(\xi,\zeta)Z^{\kappa}\{\mathbf{b}(\xi)(X)\}\frac{\mathrm{d}\xi^{2}}{\xi^{2}}, \tag{1.11}$$

where Γ goes around $\xi^2 = 1$. In particular,

$$\rho(\zeta) = \tfrac{1}{2}Z^{\kappa}\{\mathbf{t}^{*}(\zeta)(q^{2\alpha S(0)})\},$$
$$\omega(\zeta,\xi) = Z^{\kappa}\{\mathbf{b}^{*}(\zeta)\mathbf{c}^{*}(\xi)(q^{2\alpha S(0)})\}.$$

They are given in terms of the eigenvalues of the transfer matrices and the Q operators, as well as other characteristics in the Matsubara direction. Their explicit formulae will be given in (2.6) and (7.2). In appendix C we shall explain that $\omega(\zeta,\xi)$ is a quantum deformation of the canonical normalized second kind differential on a hyperelliptic Riemann surface.

From equations (1.9)–(1.11) one immediately derives

$$Z^{\kappa}\left\{\mathbf{t}^{*}\left(\zeta_1^0\right)\cdots\mathbf{t}^{*}\left(\zeta_k^0\right)\mathbf{b}^{*}\left(\zeta_1^{+}\right)\cdots\mathbf{b}^{*}\left(\zeta_l^{+}\right)\mathbf{c}^{*}(\zeta_l^{-})\cdots\mathbf{c}^{*}(\zeta_1^{-})(q^{2\alpha S(0)})\right\}$$
$$= \prod_{p=1}^{k}2\rho\left(\zeta_p^0\right)\times\det(\omega(\zeta_i^{+},\zeta_j^{-}))_{i,j=1,\ldots,l}. \tag{1.12}$$

Taking the Taylor coefficients in $\left(\zeta_i^{\epsilon}\right)^2 - 1$ in both sides, one obtains the value of Z^{κ} on an arbitrary element of the basis (1.8). This is the main result of the paper.

The text is organized as follows. In section 2, we give the precise definition of the linear functional Z^{κ} on the space $\mathcal{W}_{\alpha,0}$. We explain that on any particular $X \in \mathcal{W}_{\alpha,0}$ this functional reduces to a finite expression. In section 3, we prove (1.9). A significant part of this section is devoted to the reduction of $Z^{\kappa}\{\mathbf{t}^{*}(\zeta)(X)\}$ to finite intervals. This is a point which is used in section 6. In section 4, we explain some simple facts about transfer matrices and Q operators in the Matsubara direction. It should be considered as preparation for the following sections. In section 5, we introduce q-deformed Abelian integrals which are constructed via eigenvalues of Q operators in the Matsubara direction. We introduce q-deformed exact forms and present the q-deformed Riemann bilinear relations. In section 6, we consider $Z^{\kappa}\{\mathbf{b}^{*}(\zeta)(X)\}$. We formulate two lemmas which are proved in appendices A and B. Informally, these lemmas say that $Z^{\kappa}\{\mathbf{b}^{*}(\zeta)(X)\}$ is a q-deformation of a normalized second kind Abelian differential in ζ, which has a prescribed singularity specified by the quasi-local operator X. In the classical limit, such a differential can be expressed using the canonical normalized second kind differential. Formula (1.10) is an analogue in the quantum case, the function $\omega(\zeta,\xi)$ playing the role of the canonical differential. In section 7, we define $\omega(\zeta,\xi)$. Using the results in section 5, we prove that it satisfies all the necessary requirements. Finally, in section 8, we prove the main theorem which states that (1.10) and (1.11) hold.

As mentioned above, appendices A and B are devoted to the proof of the technical lemmas in section 6. In appendix C, we consider the classical limit of the q-deformed Abelian integrals and differentials. Then we explain that the classical limit of $\omega(\zeta,\xi)$ is indeed related to the canonical normalized second kind differential. Some general information about differentials on Riemann surfaces is provided. Readers who are not familiar with Riemann surfaces are recommended to read section 5 and appendix C together. In appendix D, we show equivalence of several non-degeneracy conditions accepted in the text.

2. Definition of the linear functional Z^{κ}

Consider a two-dimensional finite lattice composed of two one-dimensional chains: the space chain and the imaginary time or the Matsubara chain. The space chain has $2l$ sites which are labelled by the letters $j = -l+1,\ldots,l$. With every site the Pauli matrices σ_j^a are associated.

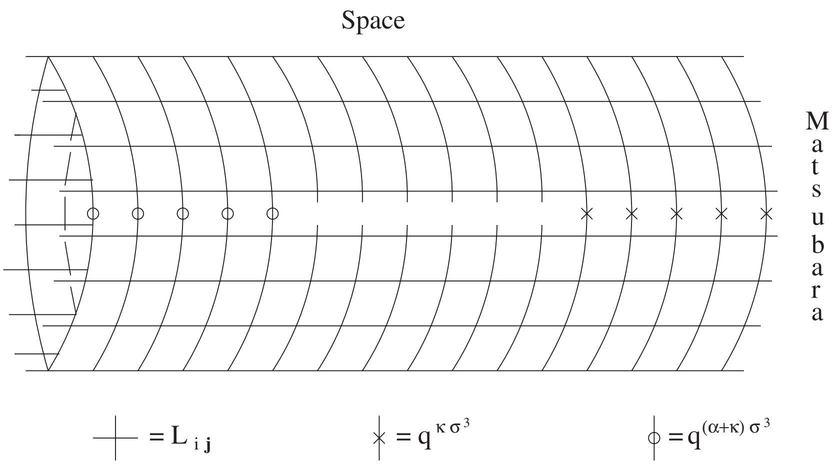

Figure 1. The broken links represent the operator $\mathcal{O}$: the arrows on them are fixed.

The Matsubara chain has $\mathbf{n}$ sites labelled by boldface letters $\mathbf{m} = \mathbf{1}, \ldots, \mathbf{n}$. With every site we associate a half-integral spin $s_{\mathbf{m}}$ and a parameter $\tau_{\mathbf{m}}$, in other words a $(2s_{\mathbf{m}} + 1)$-dimensional evaluation representation of the quantum group $U_q(\widehat{\mathfrak{sl}}_2)$. We assume that $\sum_{\mathbf{m}=1}^{\mathbf{n}} s_{\mathbf{m}}$ is an integer.

We define the monodromy matrix

$$T_{j,\mathbf{M}}(\zeta) = L_{j,\mathbf{n}}(\zeta/\tau_{\mathbf{n}})L_{j,\mathbf{n-1}}(\zeta/\tau_{\mathbf{n-1}})\cdots L_{j,\mathbf{1}}(\zeta/\tau_{\mathbf{1}}).$$

The L operator $L_{j,\mathbf{m}}(\zeta/\tau_{\mathbf{m}})$ is obtained from the universal one

$$L_j(\zeta) = q^{\frac{1}{2}} \begin{pmatrix} \zeta^2 q^{\frac{H+1}{2}} - q^{-\frac{H+1}{2}} & (q-q^{-1})\zeta F q^{\frac{H-1}{2}} \\ (q-q^{-1})\zeta q^{-\frac{H-1}{2}} E & \zeta^2 q^{-\frac{H-1}{2}} - q^{\frac{H-1}{2}} \end{pmatrix}_j ,$$

by letting E, F, H act on the $(2s_{\mathbf{m}} + 1)$-dimensional representation of $U_q(\mathfrak{sl}_2)$. We shall consider a twisted transfer matrix

$$T_{\mathbf{M}}(\zeta,\kappa) = \mathrm{Tr}_j(T_{j,\mathbf{M}}(\zeta,\kappa)),$$
$$T_{j,\mathbf{M}}(\zeta,\kappa) = T_{j,\mathbf{M}}(\zeta)q^{\kappa\sigma_j^3},$$

and use the letter $T(\zeta,\kappa)$ to denote its eigenvalues.

Now we are ready to introduce the main object of our study. On the space $\mathcal{W}_{\alpha,0}$ consider the linear functional

$$Z^{\kappa}\{q^{2\alpha S(0)}\mathcal{O}\} = \lim_{l\to\infty} \frac{\mathrm{Tr}_{\mathbf{M}}\,\mathrm{Tr}_{[-l+1,l]}\left(T_{[-l+1,l],\mathbf{M}}q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})}\mathcal{O}\right)}{\mathrm{Tr}_{\mathbf{M}}\,\mathrm{Tr}_{[-l+1,l]}\left(T_{[-l+1,l],\mathbf{M}}q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})}\right)}. \tag{2.1}$$

Here and for later use, we set

$$T_{[k,m],\mathbf{M}} = T_{k,\mathbf{M}}\cdots T_{m,\mathbf{M}}, \qquad T_{j,\mathbf{M}} = T_{j,\mathbf{M}}(1). \tag{2.2}$$

In terms of the equivalent six-vertex model, functional (2.1) is given by the partition function on the infinite cylinder (figure 1).

Suppose that the transfer matrix $T_{\mathbf{M}}(1,\kappa)$ has a unique eigenvector $|\kappa\rangle$ such that the corresponding eigenvalue $T(1,\kappa)$ has the maximal absolute value. Similarly let $\langle\kappa+\alpha|$ is be eigencovector of $T_{\mathbf{M}}(1,\kappa+\alpha)$ with the eigenvalue $T(1,\kappa+\alpha)$ possessing the same property.

Let us remark that for the XXX model the spectrum in spin zero sector is known to be simple even in the homogeneous case [14]. Suppose also that

$$\langle\kappa+\alpha|\kappa\rangle \neq 0. \tag{2.3}$$

It is clear that in this situation (2.1) reduces to the linear functional

$$Z^{\kappa}\{q^{2\alpha S(0)}\mathcal{O}\} = \lim_{l\to\infty} \frac{\langle\kappa+\alpha|\mathrm{Tr}_{[-l+1,l]}(T_{[-l+1,l],\mathbf{M}}\, q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})}\mathcal{O})|\kappa\rangle}{\langle\kappa+\alpha|\mathrm{Tr}_{[-l+1,l]}(T_{[-l+1,l],\mathbf{M}}\, q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})})|\kappa\rangle}. \tag{2.4}$$

This is the object which we shall calculate. For any given quasi-local operator we can proceed further. Indeed, if the support of $q^{2\alpha S(0)}\mathcal{O} = q^{2\alpha S(k-1)}X_{[k,m]}$ is contained in the interval $[k, m]$ of the space chain, then

$$Z^{\kappa}\{q^{2\alpha S(k-1)}X_{[k,m]}\} = \rho(1)^{k-1}\frac{\langle\kappa+\alpha|\mathrm{Tr}_{[k,m]}(T_{[k,m],\mathbf{M}}\, q^{2\kappa S_{[k,m]}}X_{[k,m]})|\kappa\rangle}{T(1,\kappa)^{m-k+1}\langle\kappa+\alpha|\kappa\rangle}, \tag{2.5}$$

where

$$\rho(\zeta) = \frac{T(\zeta,\alpha+\kappa)}{T(\zeta,\kappa)}. \tag{2.6}$$

Function (2.6) will play an important role for us; we shall see in the following section that this is the same function as in (1.9). The last formula (2.5) shows, as it has been said, that the limit $l \to \infty$ is superfluous. It is put in formula (2.4) just for the sake of treating all quasi-local operators simultaneously.

It may look surprising that the thermodynamic limit in this approach is so simple. Usually, it requires a complicated analysis of Bethe equations. Certainly, the complexity of the problem cannot disappear, and it is hidden in the limiting process $\mathbf{n} \to \infty$ to arrive at (1.3). But the idea used in [7], and developed further in [8], is that one can proceed rather far before taking this limit. This is especially true in the present work. The complexities of the thermodynamic limit of $Z^{\kappa}(X)$ are confined to only two functions, for which one can take the limit $\mathbf{n} \to \infty$ rather easily.

Let us emphasize one point which may be a source of confusion. We started with (2.1), reduced it to (2.4) and further to (2.5). The expression on the right-hand side of (2.4) is perfectly well defined for any pair of eigenvectors of $T_{\mathbf{M}}(\zeta,\kappa)$ and $T_{\mathbf{M}}(\zeta,\kappa+\alpha)$ satisfying condition (2.3). For the computation of (2.4) we shall use only quite general facts concerning Bethe vectors, so they are valid in general. Still, the subject of our study is (2.1), and it reduces to (2.4) only for the eigenvectors corresponding to the maximal eigenvalues.

3. Computation of $Z^{\kappa}\{\mathbf{t}^*(\zeta)(X)\}$

According to (1.7) we are actually interested only in the following block of $\mathbf{t}^*(\zeta)$:

$$\mathbf{t}^*(\zeta,\alpha) = \mathbf{t}^*(\zeta)|_{\mathcal{W}_{\alpha,0}\to\mathcal{W}_{\alpha,0}}.$$

Let us recall the definition of the operator $\mathbf{t}^*(\zeta,\alpha)$ given in the paper [2]. We start with a finite interval and an operator $X_{[k,m]}$. With this notation we imply that $X_{[k,m]}$ acts as I outside $[k, m]$. Define for $l > m$

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]}) = \mathrm{Tr}_a(\mathbb{T}_{a,[k,l]}(\zeta,\alpha)(X_{[k,m]})),$$

where

$$\mathbb{T}_{a,[k,l]}(\zeta,\alpha)(X_{[k,m]}) = T_{a,[k,l]}(\zeta)q^{\alpha\sigma_a^3}X_{[k,m]}T_{a,[k,l]}(\zeta)^{-1},$$
$$T_{a,[k,l]}(\zeta) = R_{a,l}(\zeta)\cdots R_{a,k}(\zeta),$$

$R_{a,j}(\zeta)$ is the standard 4×4 R-matrix (see e.g. (2.4), [2]). Define further

$$\widetilde{\mathbb{R}}^{\vee}_{i,j}(\zeta^2)=\zeta^{\mathbb{S}_i}\mathbb{R}_{i,j}(\zeta)\mathbb{P}_{i,j}\zeta^{-\mathbb{S}_j}=1+(\zeta^2-1)\mathbf{r}_{i,j}(\zeta^2),$$

where $\mathbb{P}_{i,j}(\cdot)=P_{i,j}(\cdot)P_{i,j}$ and $P_{i,j}$ is the permutation operator. Since $\widetilde{\mathbb{R}}^{\vee}_{i,j}(1)=1$, $\mathbf{r}_{i,j}(\zeta^2)$ is regular at $\zeta^2=1$. Then [2]

$$\begin{aligned}\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})&=2\sum_{j=m}^{l-1}(\zeta^2-1)^{j-m}\mathbf{r}_{j+1,j}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})\\&\quad+(\zeta^2-1)^{l-m}\,\mathrm{Tr}_a\{\mathbf{r}_{a,l}(\zeta^2)\mathbf{r}_{l,l-1}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})\},\end{aligned}$$

where $Y_{[k,m+1]}=q^{\alpha\sigma^3_k}\boldsymbol{\tau}(X_{[k,m]})$, $\boldsymbol{\tau}$ is the shift by one site of the chain to the right, and

$$\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})=\widetilde{\mathbb{R}}^{\vee}_{m+1,m}(\zeta^2)\cdots\widetilde{\mathbb{R}}^{\vee}_{k+1,k}(\zeta^2)(Y_{[k,m+1]}).$$

Hence the limit $l\to\infty$ is well defined as a power series in ζ^2-1:

$$\begin{aligned}\mathbf{t}^*(\zeta,\alpha)(q^{2\alpha S(k-1)}X_{[k,m]})&=\lim_{l\to\infty}q^{2\alpha S(k-1)}\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})\\&=2q^{2\alpha S(k-1)}\sum_{j=m}^{\infty}(\zeta^2-1)^{j-m}\mathbf{r}_{j+1,j}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]}).\end{aligned}$$

We repeated these definitions because we want to make clear the following point. Take a 2×2 matrix K such that $\mathrm{Tr}(K)\neq 0$ and consider the following object:

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha,K)(X_{[k,m]})=\frac{2}{\mathrm{Tr}(K)}\mathrm{Tr}_a(K_a\mathbb{T}_{a,[k,l]}(\zeta,\alpha)(X_{[k,m]})),$$

Then it is easy to conclude from the above definition that

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha,K)(X_{[k,m]})=\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})\qquad\mathrm{mod}\quad(\zeta^2-1)^{l-m}.\tag{3.1}$$

Lemma 3.1. *We have*

$$Z^{\kappa}\{\mathbf{t}^*(\zeta)(q^{2\alpha S(0)}\mathcal{O})\}=2\rho(\zeta)Z^{\kappa}\{q^{2\alpha S(0)}\mathcal{O}\}.\tag{3.2}$$

Proof. Without loss of generality, let $\mathcal{O}=X_{[1,m]}$ be localized on the interval $[1,m]$,

$$\begin{aligned}&Z^{\kappa}\{\mathbf{t}^*(\zeta,\alpha)(X_{[1,m]}q^{2\alpha S(0)})\}\\&\quad=\lim_{l\to\infty}\frac{\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)(X_{[1,m]}))|\kappa\rangle}{T(1,\kappa)^l\langle\kappa+\alpha|\kappa\rangle}.\end{aligned}$$

From the considerations above we obtain

$$\begin{aligned}&\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}\mathbb{T}_{a,[1,l]}(\zeta)(X_{[1,m]}))|\kappa\rangle\\&\quad=\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)(X_{[1,m]}))|\kappa\rangle\\&\quad\quad\mathrm{mod}\quad(\zeta^2-1)^{l-m}.\end{aligned}$$

The idea here is exactly as in (3.1). The monodromy matrix $T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}$ plays the role of K_a. The fact that it carries the additional structure as operator in the Matsubara space is not important. What is important is that the state $|\kappa\rangle$ is an eigenstate of $\mathrm{Tr}_a\big(T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}\big)$ with eigenvalue $T(\zeta,\kappa)$. Now we can proceed using the Yang–Baxter equation and the cyclicity of trace:

$$\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\big(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma_a^3}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)(X_{[1,m]})\big)|\kappa\rangle$$
$$=\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}(\mathbb{T}_{a,[1,l]}(\zeta)(T_{a,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma_a^3}T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}X_{[1,m]}))|\kappa\rangle$$
$$=\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}(T_{a,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma_a^3}T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}X_{[1,m]})|\kappa\rangle$$
$$=2\rho(\zeta)\langle\kappa+\alpha|\mathrm{Tr}_{[1,l]}(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}X_{[1,m]})|\kappa\rangle,$$

which proves the assertion. □

Some comments on (3.2) have to be made. It has been said that $\boldsymbol{\tau}=\mathbf{t}_1^*/2$ is the shift by one site of the chain to the right. According to [2] the rest of $\mathbf{t}_p^*$ is constructed from the adjoint action of local integrals of motion. Then, looking at (2.1) one may wonder where $\rho(\zeta)$ comes from. Naively, it should not be on the right-hand side because $\boldsymbol{\tau}$ and adjoints of the integrals of motion commute with $\mathrm{Tr}_{\mathbf{M}}(T_{Q,\mathbf{M}})$ and hence they should not contribute to (2.1) due to the cyclicity of trace. However, this is not correct in the presence of the disorder field $q^{2\alpha S(0)}$. Let us explain this point in the simplest case $\boldsymbol{\tau}=\mathbf{t}_1^*/2$. Consider definition (2.1). For finite l in (2.1), we define the cyclic shift by one site $\boldsymbol{\tau}^{\mathrm{periodic}}$, which acts in particular as

$$\boldsymbol{\tau}^{\mathrm{periodic}}(q^{2\alpha S_{[-l+1,0]}})=q^{2\alpha S_{[-l+2,1]}}.$$

On the other hand, it is easy to see from the definition that our operator $\boldsymbol{\tau}$ acts as

$$\boldsymbol{\tau}(q^{2\alpha S_{[-l+1,0]}})=q^{2\alpha S_{[-l+1,1]}}.$$

This difference accounts for the appearance of $\rho(1)$ in functional (2.1). A similar thing happens with the adjoint action of the local integral of motion $\mathbb{I}_p(\cdot)=[I_p,\cdot]$. The operator $\mathbb{I}_p^{\mathrm{periodic}}$ feels the two inhomogeneities of $q^{2\alpha S_{[-l+1,0]}}$: between sites 0 and 1 and between sites $-l+1$ and l, while the operators entering the definition of $\mathbf{t}^*(\zeta)$ feel only the first one. This is the reason why $\rho(\zeta)$ appears. There are two cases when $\rho(\zeta)=1$. The first one is trivial: $\alpha=0$. The second one is the case of VEVs (1.2) which was considered in [1, 2].

Before proceeding to $\mathbf{b}^*$ and $\mathbf{c}^*$ we have to give some explanation about q-deformed Abelian integrals.

4. Spectral properties in Matsubara direction

Consider the transfer matrix

$$T_{\mathbf{M}}(\zeta,\lambda)=\mathrm{Tr}_a\big(T_{a,\mathbf{M}}(\zeta)q^{\lambda\sigma_a^3}\big).$$

Let us introduce the Q operator

$$Q_{\mathbf{M}}^+(\zeta,\lambda)=\zeta^{\lambda-\mathbf{S}}\,\mathrm{Tr}_A(T_{A,\mathbf{M}}(\zeta,\lambda)),$$

where $\mathbf{S}$ is the total spin operator acting on the Matsubara chain, and

$$T_{A,\mathbf{M}}(\zeta,\lambda)=L_{A,\mathbf{n}}(\zeta/\tau_{\mathbf{n}})\cdots L_{A,\mathbf{1}}(\zeta/\tau_{\mathbf{1}})q^{2\lambda D_A}.$$

Here the L operators are associated with the q-oscillator algebra with generators $\mathbf{a}_A$, $\mathbf{a}_A^*$, D_A. For the notation and conventions, see [2]. If $s_{\mathbf{m}}=1/2$, then

$$L_{A,\mathbf{m}}(\zeta)=\begin{pmatrix}1-\zeta^2q^{2D_A+2} & -\zeta\mathbf{a}_A\\ -\zeta\mathbf{a}_A^* & 1\end{pmatrix}_{\mathbf{m}}\begin{pmatrix}q^{-D_A} & 0\\ 0 & q^{D_A}\end{pmatrix}_{\mathbf{m}}. \tag{4.1}$$

To obtain $L_{A,\mathbf{m}}(\zeta)$ for other spins, one applies the standard fusion procedure. The Q operator $Q^-_{\mathbf{M}}(\zeta,\lambda)$ is defined by

$$Q^-_{\mathbf{M}}(\zeta,\lambda)=JQ^+_{\mathbf{M}}(\zeta,-\lambda)J,$$

where J is the operator of spin reversal.

These Q operators satisfy the Baxter equation:

$$T_{\mathbf{M}}(\zeta,\lambda)Q^{\pm}_{\mathbf{M}}(\zeta,\lambda)=d(\zeta)Q^{\pm}_{\mathbf{M}}(\zeta q,\lambda)+a(\zeta)Q^{\pm}_{\mathbf{M}}(\zeta q^{-1},\lambda). \tag{4.2}$$

These equations hold for the eigenvalues because $T_{\mathbf{M}}(\zeta,\lambda)$ commute with $Q^{\pm}_{\mathbf{M}}(\xi,\lambda)$.

The functions $a(\zeta), d(\zeta)$ are defined by spins and inhomogeneities present in the Matsubara direction

$$\begin{aligned} a(\zeta)&=\prod_{\mathbf{m}=1}^{\mathbf{n}}a_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}), \qquad & a_s(\zeta)&=\zeta^2q^{2s+1}-1,\\ d(\zeta)&=\prod_{\mathbf{m}=1}^{\mathbf{n}}d_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}), \qquad & d_s(\zeta)&=\zeta^2q^{-2s+1}-1. \end{aligned} \tag{4.3}$$

Let us cite one formula from [5]

$$Q^+_{\mathbf{M}}(\zeta,\lambda)Q^-_{\mathbf{M}}(\zeta q,\lambda)-Q^-_{\mathbf{M}}(\zeta,\lambda)Q^+_{\mathbf{M}}(\zeta q,\lambda)=\frac{1}{q^{\lambda-\mathbf{S}}-q^{-\lambda+\mathbf{S}}}W(\zeta), \tag{4.4}$$

where

$$W(\zeta)=\prod_{\mathbf{m}=1}^{\mathbf{n}}w_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}), \qquad w_s(\zeta)=\prod_{k=1}^{2s}(1-\zeta^2q^{2k-2s+1}).$$

Suppose that $T_{\mathbf{M}}(\zeta,\lambda)$ has a unique eigenvector $|\lambda\rangle$ with eigenvalue $T(\zeta,\lambda)$ such that $T(1,\lambda)$ has maximal absolute value. We denote by $Q^{\pm}(\xi,\lambda)$ the eigenvalues of $Q^{\pm}_{\mathbf{M}}(\xi,\lambda)$ on $|\lambda\rangle$. If the eigenvector $|\lambda\rangle$ has spin $d-\sum_{\mathbf{m}=1}^{\mathbf{n}}s_{\mathbf{m}}$, it follows from the form of the L operator (4.1) that $\zeta^{-\lambda+\mathbf{S}}Q^+(\zeta,\lambda)$ is a polynomial in ζ^2 of degree d, while $\zeta^{\lambda-\mathbf{S}}Q^-(\zeta,\lambda)$ is of degree $2\sum_{\mathbf{m}=1}^{\mathbf{n}}s_{\mathbf{m}}-d$. Due to the quantum Wronskian relation (4.4), their leading and the lowest coefficients are both nonzero.

Let us discuss the symmetry under negating λ. We have

$$T_{\mathbf{M}}(\zeta,-\lambda)=JT_{\mathbf{M}}(\zeta,\lambda)J, \tag{4.5}$$

which implies that the spectra of $T_{\mathbf{M}}(\zeta,\lambda)$ and $T_{\mathbf{M}}(\zeta,-\lambda)$ coincide, and, in particular,

$$T(\zeta,\lambda)=T(\zeta,-\lambda). \tag{4.6}$$

Furthermore, the equation

$$Q^-_{\mathbf{M}}(\zeta,\lambda)=JQ^+_{\mathbf{M}}(\zeta,-\lambda)J$$

implies that

$$Q^-(\zeta,\lambda)=Q^+(\zeta,-\lambda). \tag{4.7}$$

Due to (4.5) the vectors $|\lambda\rangle$ and $|-\lambda\rangle$ have opposite spins.

5. Deformed Abelian integrals

Working with quantum integrable models, one should not neglect the important piece of intuition provided by the method of separation of variables (SoV) discovered by Sklyanin [13]. It has been explained in [10] that the matrix elements of observables in the SoV method are expressed in terms of deformed Abelian integrals. In the case under consideration, which is related to the algebra $U_q(\widehat{\mathfrak{sl}}_2)$, these integrals are deformations of hyperelliptic ones. Let us give their definition.

Introduce the function $\varphi(\zeta)$ which satisfies the equation

$$a(\zeta q)\varphi(\zeta q) = d(\zeta)\varphi(\zeta). \tag{5.1}$$

This function is elementary

$$\varphi(\zeta) = \prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}} \varphi_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}}), \qquad \varphi_s(\zeta) = \prod_{k=0}^{2s} \frac{1}{\zeta^2 q^{-2s+2k+1} - 1}.$$

In addition to the contour Γ which encircles $\zeta^2 = 1$, we consider $\mathbf{n+1}$ contours in the ζ^2 plane: $\Gamma_{\mathbf{0}}$ which goes around 0, and $\Gamma_{\mathbf{m}}$ which encircles the poles $\zeta^2 = \tau_{\mathbf{m}}^2 q^{2s_{\mathbf{m}}-2k-1}$ $(k = 0, \ldots, 2s_{\mathbf{m}})$ of $\varphi_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}})$.

In the following, we use the q-difference operator

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1}).$$

It acts on the class of functions of the form $f(\zeta) = \zeta^\lambda P(\zeta^2)$, P being a polynomial in ζ^2 and $q^{2(n+\lambda)} \neq 1$ for all integers n. Within this class the q-primitive $\Delta_\zeta^{-1} f(\zeta)$ is defined uniquely.

There are two kinds of deformed Abelian integrals,

$$\int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta) Q^{\mp}(\zeta, \kappa+\alpha) Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}, \tag{5.2}$$

where $\zeta^{\mp\alpha} f^{\pm}(\zeta)$ is a polynomial in ζ^2, in order that the integrand is single valued.

We start our study of deformed Abelian integrals with the following technical lemma.

Lemma 5.1. *Let $\zeta^{\mp\alpha} f^{\pm}(\zeta)$ be a polynomial in ζ^2. Then, for $\mathbf{m} = \mathbf{0}, \ldots, \mathbf{n}$, the following identities hold:*

$$\int_{\Gamma_{\mathbf{m}}} \left\{T(\zeta,\kappa)\Delta_\zeta^{-1} f^{\pm}(\zeta q) - T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1} f^{\pm}(\zeta)\right\} Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}$$
$$= \int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta) a(\zeta) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta q^{-1},\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}, \tag{5.3}$$

$$\int_{\Gamma_{\mathbf{m}}} \left\{T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1} f^{\pm}(\zeta) - T(\zeta,\kappa)\Delta_\zeta^{-1} f^{\pm}(\zeta q^{-1})\right\} Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}$$
$$= \int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta) d(\zeta) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta q,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}. \tag{5.4}$$

Proof. This can be verified directly by applying the Baxter equation to $T(\zeta,\kappa)Q^{\pm}(\zeta,\kappa)$, $T(\zeta,\kappa+\alpha)Q^{\mp}(\zeta,\kappa+\alpha)$ and moving contours of integration. □

It is well known that, on a compact Riemann surface of genus g, the space of the first and the second kind differentials (meromorphic differentials without residues) has a finite dimension $2g$ when considered modulo exact forms. In [10] it was explained what are the

q-deformed exact forms, for which the deformed Abelian integrals vanish. Since the proof was omitted in that paper we include it here.

Lemma 5.2. *Define a q-deformed exact form to be an expression*

$$\begin{aligned}E(f^{\pm}(\zeta)) = {} & T(\zeta,\kappa)\Delta_\zeta^{-1}(f^{\pm}(\zeta)T(\zeta,\kappa)) + T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}(f^{\pm}(\zeta)T(\zeta,\kappa+\alpha))\\ & - T(\zeta,\kappa)\Delta_\zeta^{-1}(f^{\pm}(\zeta q)T(\zeta q,\kappa+\alpha))\\ & - T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}(f^{\pm}(\zeta q^{-1})T(\zeta q^{-1},\kappa))\\ & + a(\zeta q)d(\zeta)f^{\pm}(\zeta q) - d(\zeta q^{-1})a(\zeta)f^{\pm}(\zeta q^{-1}),\end{aligned} \tag{5.5}$$

where $\zeta^{\mp\alpha}f^{\pm}(\zeta)$ is a polynomial in ζ^2. Then we have

$$\int_{\Gamma_{\mathbf{m}}} E(f^{\pm}(\zeta))Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = 0.$$

Proof. Let us divide the integral into two pieces

$$\int_{\Gamma_{\mathbf{m}}} E(f^{\pm}(\zeta))Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = I_1 + I_2, \tag{5.6}$$

where I_1 contains first four terms from (5.5) and I_2 contains remaining two. Apply (5.3) to the first and the fourth terms in I_1, and to the second and the third terms as well. Using the Baxter equation and moving contours using (5.1), one obtains

$$\begin{aligned}I_1 = {} & \int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta q^{-1})T(\zeta q^{-1},\kappa)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta q^{-1},\kappa)a(\zeta)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}\\ & - \int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta)T(\zeta,\kappa+\alpha)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta q^{-1},\kappa)a(\zeta)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}.\end{aligned}$$

Now apply the Baxter equation

$$\begin{aligned}I_1 = {} & \int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta q^{-1})Q^{\mp}(\zeta,\kappa+\alpha)(Q^{\pm}(\zeta q^{-2},\kappa)a(\zeta q^{-1}) + Q^{\pm}(\zeta,\kappa)d(\zeta q^{-1}))a(\zeta)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}\\ & - \int_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta)(a(\zeta)Q^{\mp}(\zeta q^{-1},\kappa+\alpha)\\ & + d(\zeta)Q^{\mp}(\zeta q,\kappa+\alpha))Q^{\pm}(\zeta q^{-1},\kappa)a(\zeta)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}.\end{aligned}$$

Moving contours we find

$$I_1 = \int_{\Gamma_{\mathbf{m}}} \{d(\zeta q^{-1})a(\zeta)f^{\pm}(\zeta q^{-1}) - a(\zeta q)d(\zeta)f^{\pm}(\zeta q)\}Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2},$$

i.e. $I_1 = -I_2$. □

A beautiful feature of deformed Abelian integrals is that they allow for a deformation of the Riemann bilinear relations as well. In [10] the latter are given in the full-fledged form. For our present purposes, it is sufficient to use a part of them given by the following lemma.

Lemma 5.3. *Consider the following function in two variables:*

$$r(\zeta,\xi) = r^{+}(\zeta,\xi) - r^{-}(\xi,\zeta),$$

where

$$r^{+}(\zeta,\xi) = r^{+}(\zeta,\xi|\kappa,\alpha), \qquad r^{-}(\xi,\zeta) = r^{+}(\xi,\zeta|-\kappa,-\alpha),$$

and

$$\begin{aligned} r^{+}(\zeta,\xi|\kappa,\alpha) = {} & T(\zeta,\kappa)\Delta_\zeta^{-1}(\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa)-T(\xi,\kappa))) \\ & + T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}(\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa+\alpha)-T(\xi,\kappa+\alpha))) \\ & - T(\zeta,\kappa)\Delta_\zeta^{-1}(\psi(q\zeta/\xi,\alpha)(T(\zeta q,\kappa+\alpha)-T(\xi,\kappa+\alpha))) \\ & - T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}(\psi(q^{-1}\zeta/\xi,\alpha)(T(\zeta q^{-1},\kappa)-T(\xi,\kappa))) \\ & + (a(\zeta q)-a(\xi))d(\zeta)\psi(q\zeta/\xi,\alpha) - (d(\zeta q^{-1})-d(\xi))a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha). \end{aligned} \tag{5.7}$$

Then

$$\int_{\Gamma_{\mathbf{i}}}\int_{\Gamma_{\mathbf{j}}} r(\zeta,\xi)Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)Q^{+}(\xi,\kappa+\alpha)Q^{-}(\xi,\kappa)\varphi(\zeta)\varphi(\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2}\frac{\mathrm{d}\xi^2}{\xi^2} = 0. \tag{5.8}$$

Proof. The proof is similar to that of the previous lemma. We apply lemma 5.1, invoke the Baxter equation and move the contours. When the Baxter equation is applied to expressions like $\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa)-T(\xi,\kappa))$ separately with respect to ζ and ξ, a singularity may appear from $\psi(\zeta/\xi,\alpha)$. In general, by moving the contours such a singularity produces intersection numbers as in the genuine Riemann bilinear relations (see [11]). In the present case, this does not happen because the contours do not have nontrivial intersections. □

Clearly $\xi^{\alpha}r^{+}(\zeta,\xi)$ is a polynomial in ξ^2 and $\zeta^{-\alpha}r^{-}(\xi,\zeta)$ is a polynomial in ζ^2, both of degree $\mathbf{n}$. This allows us to define the polynomials $p^{\pm}_{\mathbf{m}}$ by

$$r^{+}(\zeta,\xi) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \zeta^{\alpha}p^{+}_{\mathbf{m}}(\zeta^2)\xi^{-\alpha+2\mathbf{m}}, \qquad r^{-}(\xi,\zeta) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \xi^{-\alpha}p^{-}_{\mathbf{m}}(\xi^2)\zeta^{\alpha+2\mathbf{m}}.$$

Introduce the $(\mathbf{n}+\mathbf{1})\times(\mathbf{n}+\mathbf{1})$ matrices

$$\mathcal{A}^{\pm}_{\mathbf{i},\mathbf{j}} = \int_{\Gamma_{\mathbf{i}}} \zeta^{\pm\alpha+2\mathbf{j}}Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}, \tag{5.9}$$

$$\mathcal{B}^{\pm}_{\mathbf{i},\mathbf{j}} = \int_{\Gamma_{\mathbf{i}}} \zeta^{\pm\alpha}p^{\pm}_{\mathbf{j}}(\zeta^2)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}. \tag{5.10}$$

Then (5.8) reads as

$$\mathcal{B}^{+}(\mathcal{A}^{-})^{t} = \mathcal{A}^{+}(\mathcal{B}^{-})^{t}. \tag{5.11}$$

We explain in appendix C that, in the classical limit $q\to 1$ (and for $\alpha=0$), $\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}$ reduce to the matrices of a-periods of differentials of the first and the second kinds, respectively. Relation (5.11) becomes one quarter of the classical Riemann bilinear relations which state that the full matrix of a- and b-periods is an element of the symplectic group.

Before closing this section let us make a comment. Suppose $\zeta^{\mp\alpha}f(\zeta)$ is a rational function. We assume that the poles of this function do not overlap those of $\varphi(\zeta)$ and $\zeta^2=0$. In this case, the q-primitive $\Delta_\zeta^{-1}f(\zeta)$ is not uniquely defined, and in general develops infinitely many poles $q^{2n}w$ ($n\in\mathbb{Z}$, w are the poles of $\zeta^{\mp\alpha}f(\zeta)$). Nevertheless lemma 5.1 remains true. Actually it tells that the deformed Abelian integrals on the left-hand side of (5.3) and (5.4) do not depend on a particular choice of the q-primitive. For the same reason, deformed Abelian integrals of the q-exact form in lemma 5.2 have unambiguous meaning. Later on we shall deal with examples of such q-primitives of the form $\Delta_\zeta^{-1}(\psi(\zeta/\xi,\alpha)P(\zeta^2))$ or $\Delta_\zeta^{-1}(\psi(\xi/\zeta,\alpha)P(\zeta^2))$.

6. Properties of $Z^\kappa\{\mathbf{b}^*(\zeta)(X)\}$ and $Z^\kappa\{\mathbf{c}^*(\zeta)X)\}$

Our strategy is to compute $Z^\kappa\{\mathbf{b}^*(\zeta)(X)\}$ and $Z^\kappa\{\mathbf{c}^*(\zeta)X)\}$ inductively, reducing them to similar quantities involving the annihilation operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$. It has been said in the Introduction that $Z^\kappa\{\mathbf{b}^*(\zeta)(X)\}$ is nontrivial only when $X \in \mathcal{W}_{\alpha+1,-1}$, and $Z^\kappa\{\mathbf{c}^*(\zeta)(X)\}$ is nontrivial only when $X \in \mathcal{W}_{\alpha-1,1}$. We denote these blocks by

$$\mathbf{b}^*(\zeta,\alpha) = \mathbf{b}^*(\zeta)|_{\mathcal{W}_{\alpha+1,-1}\to\mathcal{W}_{\alpha,0}}, \qquad \mathbf{c}(\zeta,\alpha) = \mathbf{c}(\zeta)|_{\mathcal{W}_{\alpha+1,-1}\to\mathcal{W}_{\alpha,0}},$$
$$\mathbf{c}^*(\zeta,\alpha) = \mathbf{c}^*(\zeta)|_{\mathcal{W}_{\alpha-1,1}\to\mathcal{W}_{\alpha,0}}, \qquad \mathbf{b}(\zeta,\alpha) = \mathbf{b}(\zeta)|_{\mathcal{W}_{\alpha-1,1}\to\mathcal{W}_{\alpha,0}}.$$

Hence the nontrivial part of our main equations (1.10) and (1.11) takes the form

$$Z^\kappa\{\mathbf{b}^*(\zeta,\alpha)(X)\} = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \omega(\zeta,\xi)Z^\kappa\{\mathbf{c}(\xi,\alpha)(X)\}\frac{\mathrm{d}\xi^2}{\xi^2}, \qquad X \in \mathcal{W}_{\alpha+1,-1}, \tag{6.1}$$

$$Z^\kappa\{\mathbf{c}^*(\zeta,\alpha)(X)\} = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \omega(\zeta,\xi)Z^\kappa\{\mathbf{b}(\xi,\alpha)(X)\}\frac{\mathrm{d}\xi^2}{\xi^2}, \qquad X \in \mathcal{W}_{\alpha-1,1}. \tag{6.2}$$

Our task is to establish the existence of $\omega(\zeta,\xi)$ and to determine it explicitly. In view of the spin reversal symmetry which relates $(\mathbf{b}^*, \mathbf{c})$ with $(\mathbf{c}^*, \mathbf{b})$, we shall concentrate on the first pair.

Apart from an overall power of ζ, $\mathbf{b}^*(\zeta,\alpha)$ is defined *a priori* as a formal power series in ζ^2-1. Nevertheless when acting on each operator it reduces to a rational function, due to the same mechanism as explained for $\mathbf{t}^*$. Namely,

$$\mathbf{b}^*(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]}) = q^{2\alpha S(0)}\lim_{l\to\infty}\mathrm{Tr}_c\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})\}. \tag{6.3}$$

We recall the definition of the operator $\mathbf{g}_{c,[1,m]}(\zeta,\alpha)$ in appendix A. Formula (6.3) together with the requirement of translational invariance can be considered as a definition of $\mathbf{b}^*(\zeta,\alpha)$, but self-consistency of this definition requires that $\mathbf{g}_{c,[1,m]}(\zeta,\alpha)$ satisfies certain reduction relations which were proved in [2].

Using (6.3) we find by the same method as in lemma 3.1

$$T(\zeta,\kappa)Z^\kappa(\mathbf{b}^*(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]})) \\ = \frac{\mathrm{Tr}_{[1,m],c}(\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa)2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})|\kappa\rangle)}{T(1,\kappa)^m\langle\kappa+\alpha|\kappa\rangle}.$$

Due to this equation the left-hand side happens to be up to the overall multiplier ζ^α a rational function of ζ^2 with poles only at $\zeta^2 = q^{\pm 2}$. Its singular part is given as follows.

Lemma 6.1. *Set*

$$\omega_{\mathrm{sing}}(\zeta,\xi) = -\Delta_\zeta\psi(\zeta/\xi,\alpha) + \frac{4}{T(\zeta,\kappa)T(\xi,\kappa)}(a(\xi)d(q^{-1}\xi)\psi(q\zeta/\xi,\alpha) \\ - a(q\xi)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha)). \tag{6.5}$$

Then we have

$$T(\zeta,\kappa)Z^\kappa\left\{\left(\mathbf{b}^*(\zeta,\alpha) - \frac{1}{2\pi\mathrm{i}}\oint_\Gamma\omega_{\mathrm{sing}}(\zeta,\xi)\mathbf{c}(\xi,\alpha)\frac{d\xi^2}{\xi^2}\right)(X)\right\} = \zeta^\alpha P_{\mathbf{n}}(\zeta^2), \tag{6.6}$$

where $X \in \mathcal{W}_{\alpha+1,-1}, \Gamma$ *encircles* $\xi^2 = 1$, *and* $P_{\mathbf{n}}(\zeta^2)$ *is a polynomial in* ζ^2 *of degree at most* $\mathbf{n}$.

Lemma 6.1 is proved in appendix A.

In order to characterize the quantity on the left-hand side of (6.6), we need to have a control over the unknown polynomial $P_{\mathbf{n}}(\zeta^2)$. This is the point where deformed Abelian integrals come into play. Introduce the notation

$$\overline{D}_\zeta F(\zeta) = F(q\zeta) + F(q^{-1}\zeta) - 2\rho(\zeta)F(\zeta).$$

Lemma 6.2. *For* $\mathbf{m} = \mathbf{0}, \ldots, \mathbf{n}$, *the following relations hold:*

$$\int_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)Z^\kappa\left\{(\mathbf{b}^*(\zeta,\alpha) + \frac{1}{2\pi \mathrm{i}}\oint_\Gamma \frac{\mathrm{d}\xi^2}{\xi^2}\left(\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\right)\mathbf{c}(\xi,\alpha))(X)\right\}$$
$$\times\, Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = 0, \qquad (6.7)$$

for $X \in \mathcal{W}_{\alpha+1,-1}$.

As explained at the end of section 5, one can apply lemma 5.1 to $f^+(\zeta) = \overline{D}_\xi\psi(\zeta/\xi,\alpha)$. Then, the integral over ζ^2 in the second term can be rewritten as

$$\int_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}$$
$$= \int_{\Gamma_{\mathbf{m}}} \overline{D}_\xi\psi(\zeta/\xi,\alpha)Q^-(\zeta,\kappa+\alpha)$$
$$\times\,(a(\zeta)Q^+(q^{-1}\zeta,\kappa) - d(\zeta)Q^+(q\zeta,\kappa))\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}. \qquad (6.8)$$

Hence it does not actually depend on a particular choice of $\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)$.

Proof of lemma 6.2 is long and technical. We defer it to appendix B.

Comparing (6.1) with (6.6) and (6.7), we infer that the function $\omega(\zeta,\xi) = \omega(\zeta,\xi|\kappa,\alpha)$ satisfy the conditions:

(1) Singular part

$$\zeta^{-\alpha}T(\zeta,\kappa)(\omega(\zeta,\xi) - \omega_{\mathrm{sing}}(\zeta,\xi)) \text{ is a polynomial in } \zeta^2 \text{ of degree } \mathbf{n}. \qquad (6.9)$$

(2) Normalization

$$\int_{\Gamma_{\mathbf{m}}} T(\zeta,\kappa)\left(\omega(\zeta,\xi) + \overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\right)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = 0, \qquad (6.10)$$

$$(\mathbf{m} = \mathbf{0}, \ldots, \mathbf{n}). \qquad (6.11)$$

Furthermore, equation (6.2) requires an additional property of $\omega(\zeta,\xi|\kappa,\alpha)$ (see section 7).

(3) Symmetry

$$\omega(\xi,\zeta|-\kappa,-\alpha) = \omega(\zeta,\xi|\kappa,\alpha). \qquad (6.12)$$

7. Definition of $\omega(\zeta,\xi)$ and its symmetry

We shall first give the definition of the function $\omega(\zeta,\xi)$, and then prove that it satisfies all the necessary properties.

In section 5, we defined the matrices $\mathcal{A}^+$ and $\mathcal{B}^+$. In appendix D we show that the condition

$$\det \mathcal{A}^+ \neq 0 \tag{7.1}$$

is equivalent to the non-degeneracy condition (2.3) accepted previously. The classical analogue of (7.1) states that 'there are no holomorphic differentials such that all the a-periods vanish'.

Assuming (7.1), consider the function

$$\omega(\zeta,\xi|\kappa,\alpha)=\frac{4}{T(\zeta,\kappa)T(\xi,\kappa)}v^+(\zeta)^t(\mathcal{A}^+)^{-1}\mathcal{B}^+v^-(\xi)+\omega_{\mathrm{sym}}(\zeta,\xi|\kappa,\alpha), \tag{7.2}$$

where $v^{\pm}(\zeta)$ are vectors with components $v^{\pm}(\zeta)_{\mathbf{j}}=\zeta^{\pm\alpha+2\mathbf{j}}$, $\mathcal{A}$, $\mathcal{B}$ are given by (5.9) and (5.10), and

$$\begin{aligned}\omega_{\mathrm{sym}}(\zeta,\xi|\kappa,\alpha)=\frac{1}{T(\zeta,\kappa)T(\xi,\kappa)}&\{(4a(\xi)d(\zeta)-T(\zeta,\kappa)T(\xi,\kappa))\psi(q\zeta/\xi,\alpha)\\ &-(4a(\zeta)d(\xi)-T(\zeta,\kappa)T(\xi,\kappa))\psi(q^{-1}\zeta/\xi,\alpha)\\ &-2\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa)T(\xi,\kappa+\alpha)-T(\xi,\kappa)T(\zeta,\kappa+\alpha))\}.\end{aligned}$$

The function $\omega_{\mathrm{sym}}(\zeta,\xi|\kappa,\alpha)$ satisfies the relation

$$\omega_{\mathrm{sym}}(\zeta,\xi|\kappa,\alpha)=\omega_{\mathrm{sym}}(\xi,\zeta|-\kappa,-\alpha) \tag{7.3}$$

due to (4.6) and the equality $\psi(\zeta^{-1},-\alpha)=-\psi(\zeta,\alpha)$.

The function $\zeta^{-\alpha}\omega(\zeta,\xi|\kappa,\alpha)$ is a rational function of ζ^2. It is clear by the construction that property (6.9) is satisfied.

The remaining properties (6.10) and (6.12) of $\omega(\zeta,\xi|\kappa,\alpha)$ are more complicated, and we formulate them as lemmas.

Lemma 7.1. *The function $\omega(\zeta,\xi|\kappa,\alpha)$ defined by (7.2) satisfies the normalization condition (6.10).*

Proof. By using definitions (5.9) and (5.10) we have

$$\begin{aligned}\int_{\Gamma_{\mathbf{m}}} & v^+(\zeta)^t(\mathcal{A}^+)^{-1}\mathcal{B}^+v^-(\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\\ &=(\mathcal{B}^+v^-(\xi))_{\mathbf{m}}\\ &=\int_{\Gamma_{\mathbf{m}}} r^+(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2},\end{aligned}$$

Definition (5.7) can be rewritten as

$$r^+(\zeta,\xi)=E(\psi(\zeta/\xi,\alpha))-\tfrac{1}{4}T(\xi,\kappa)T(\zeta,\kappa)\big\{\omega_{\mathrm{sym}}(\zeta,\xi|\kappa,\alpha)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\big\}.$$

Therefore, the normalization condition (6.10) follows from

$$\int_{\Gamma_{\mathbf{m}}} E(\psi(\zeta/\xi,\alpha))Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0.$$

□

Lemma 7.2. *The function $\omega(\zeta,\xi|\kappa,\alpha)$ defined by (7.2) satisfies the symmetry condition (6.12).*

Proof. In section 5, we had relation (5.11)

$$\mathcal{B}^+(\mathcal{A}^-)^t = \mathcal{A}^+(\mathcal{B}^-)^t. \tag{7.4}$$

In appendix D we show that $\det(\mathcal{A}^-) \neq 0$ follows from condition (2.3). Hence both $\mathcal{A}^\pm$ can be inverted. So, invert them and multiply the result by $v^+(\zeta)^t$ from the left and $v^-(\xi)$ from the right:

$$v^+(\zeta)^t(\mathcal{A}^+)^{-1}\mathcal{B}^+ v^-(\xi) = v^-(\xi)^t(\mathcal{A}^-)^{-1}\mathcal{B}^- v^+(\zeta).$$

What remains to do is to add $\omega_{\mathrm{sym}}(\zeta,\xi|\kappa,\alpha)$ to both sides, to use (7.3) and to recall identities (4.6) and (4.7). □

8. Main theorem

Now we are able to prove our main theorem.

Theorem 8.1. *Under the generality condition (2.3) we have*

$$Z^\kappa\{\mathbf{b}^*(\zeta)(X)\} = \frac{1}{2\pi \mathrm{i}}\oint_\Gamma \omega(\zeta,\xi)Z^\kappa\{\mathbf{c}(\xi)(X)\}\frac{\mathrm{d}\xi^2}{\xi^2}, \tag{8.1}$$

$$Z^\kappa\{\mathbf{c}^*(\zeta)(X)\} = -\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \omega(\xi,\zeta)Z^\kappa\{\mathbf{b}(\xi)(X)\}\frac{\mathrm{d}\xi^2}{\xi^2}. \tag{8.2}$$

Proof. Consider (8.1). It has been said that it is sufficient to consider the blocks $\mathbf{b}^*(\zeta,\alpha)$, $\mathbf{c}(\xi,\alpha)$. Due to the structure of singularities (6.6) and (6.9) we have

$$T(\zeta,\kappa)Z^\kappa\left\{(\mathbf{b}^*(\zeta,\alpha) - \frac{1}{2\pi \mathrm{i}}\oint_\Gamma \omega(\zeta,\xi)\mathbf{c}(\xi,\alpha))(X)\right\}\frac{\mathrm{d}\xi^2}{\xi^2} = \zeta^\alpha \tilde{P}_{\mathbf{n}}(\zeta^2), \tag{8.3}$$

where $P_{\mathbf{n}}(\zeta^2)$ is a polynomial of degree $\mathbf{n}$. Due to lemma 6.2 and lemma 7.1 we have

$$\int_{\Gamma_{\mathbf{m}}} \zeta^\alpha P_{\mathbf{n}}(\zeta^2)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = 0, \qquad \mathbf{m} = \mathbf{0},\dots,\mathbf{n},$$

which implies $P_{\mathbf{n}}(\zeta^2) = 0$ due to (7.1).

Now consider (8.2). According to [2], the operators $\mathbf{c}^*, \mathbf{b}$ are related to $\mathbf{b}^*, \mathbf{c}$ by the transformation

$$\phi_\alpha(\mathbf{x}(\zeta,\alpha)) = q^{-1}N(\alpha-1)\circ\mathbb{J}\circ\mathbf{x}(\zeta,-\alpha)\circ\mathbb{J},$$

where $N(x) = q^{-x} - q^x$ and $\mathbb{J}(X) = JXJ^{-1}$ is the spin reversal. Namely,

$$\mathbf{c}^*(\zeta,\alpha) = -\phi_\alpha(\mathbf{b}^*(\zeta,\alpha)), \qquad \mathbf{b}(\zeta,\alpha) = \phi_\alpha(\mathbf{c}(\zeta,\alpha)).$$

It is also easy to see that

$$Z^\kappa\{X\} = Z^{-\kappa}\{\mathbb{J}(X)\}.$$

Hence (8.1) implies

$$Z^\kappa\{\mathbf{c}^*(\zeta,\alpha)(X)\} = -\frac{1}{2\pi \mathrm{i}}\oint_\Gamma \omega(\zeta,\xi|-\kappa,-\alpha)Z^\kappa\{\mathbf{b}(\xi,\alpha)(X)\}\frac{q\xi^2}{\xi^2},$$

which is equivalent to (8.2) due to (6.12). □

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Appendix A. Proof of lemma 6.1

In this appendix, we prove lemma 6.1. We also prove some additional result used in appendix B (corollary A.2). First let us comment on equation (6.4). This formula is used in the proof in order to reduce the action of the operator $\mathbf{b}^*$, when it is considered inside the functional Z^κ, to that of an operator on the interval $[1, m]$. This is a great simplification, because without Z^κ the support of the coefficients in the expansion of $\mathbf{b}^*(\zeta, \alpha)$ in $\zeta^2 - 1$ becomes indefinitely large. In other words, inside Z^κ the series expansion of $\mathbf{b}^*(\zeta, \alpha)$ can be summed up to a rational function. Therefore, the proof of lemma 6.1 consists of computing the singular part of the rational function. This task is done indirectly by considering an inhomogeneous space chain. We introduce inhomogeneity parameters $\boldsymbol{\xi} = (\xi_j)$, so that the original multiple poles $\zeta^2 = q^{\pm 2}$ in the homogeneous chain are split into simple poles $\zeta^2 = q^{\pm 2}\xi_j^2$ for $1 \leqslant j \leqslant m$. Define the functional

$$Z^\kappa_{[1,m]}\{X_{[1,m]}\} = \frac{\mathrm{Tr}_{[1,m]}\langle \kappa + \alpha | T_{[1,m],\mathbf{M}}(\boldsymbol{\xi}, \kappa) X_{[1,m]} | \kappa\rangle}{\prod_{j=1}^m T(\xi_j, \kappa)\langle \kappa + \alpha | \kappa\rangle},$$
$$T_{[1,m],\mathbf{M}}(\boldsymbol{\xi}, \kappa) = T_{1,\mathbf{M}}(\xi_1, \kappa) \cdots T_{m,\mathbf{M}}(\xi_m, \kappa). \tag{A.1}$$

Using this functional equation (6.4) takes the form

$$Z^\kappa\{\mathbf{b}^*(\zeta, \alpha)(q^{2(\alpha+1)S(0)} X_{[1,m]})\}$$
$$= Z^\kappa_{[1,m+1]}\{2\mathbf{g}_{m+1,[1,m]}(\xi_{m+1}, \alpha)(X_{[1,m]})\}|_{\xi_1 = \cdots = \xi_m = 1, \xi_{m+1} = \zeta}. \tag{A.2}$$

First recall from [2] the definition of the operator $\mathbf{k}_{[1,m]}(\zeta, \alpha)$ and its basic relations with $\mathbf{c}_{[1,m]}(\zeta, \alpha)$, $\bar{\mathbf{c}}_{[1,m]}(\zeta, \alpha)$, $\mathbf{f}_{[1,m]}(\zeta, \alpha)$:

$$\mathbf{k}_{[1,m]}(\zeta, \alpha)(X_{[1,m]}) = \mathrm{Tr}_{a,A}\left\{\sigma_a^+ \mathbb{T}_{\{a,A\},[1,m]}(\zeta, \alpha)\zeta^{\alpha - \mathbb{S}_{[1,m]}}(q^{-2S_{[1,m]}} X_{[1,m]})\right\}, \tag{A.3}$$

$$\mathbf{k}_{[1,m]}(\zeta, \alpha)(X_{[1,m]}) - \Delta_\zeta \mathbf{f}_{[1,m]}(\zeta, \alpha)(X_{[1,m]})$$
$$= \mathbf{c}_{[1,m]}(q\zeta, \alpha)(X_{[1,m]}) + \mathbf{c}_{[1,m]}(q^{-1}\zeta, \alpha)(X_{[1,m]}) + \bar{\mathbf{c}}_{[1,m]}(\zeta, \alpha)(X_{[1,m]}). \tag{A.4}$$

In the complex plane the operators $\mathbf{c}_{[1,m]}(\zeta, \alpha)$, $\bar{\mathbf{c}}_{[1,m]}(\zeta, \alpha)$, $\mathbf{f}_{[1,m]}(\zeta, \alpha)$ have singularities at $\zeta^2 = \xi_j^2$ only. The operator $\mathbf{g}_{c,[1,m]}(\zeta, \alpha)$ is given by

$$2\mathbf{g}_{c,[1,m]}(\zeta, \alpha)(X_{[1,m]}) = \mathbf{f}_{[1,m]}(q\zeta, \alpha)(X_{[1,m]}) + \mathbf{f}_{[1,m]}(q^{-1}\zeta, \alpha)(X_{[1,m]})$$
$$- 2\mathbb{T}_{c,[1,m]}(\zeta, \alpha)\mathbf{f}_{[1,m]}(\zeta, \alpha)(X_{[1,m]}) + 2\mathbf{u}_{c,[1,m]}(\zeta, \alpha)(X_{[1,m]}), \tag{A.5}$$

where

$$\mathbf{u}_{c,[1,m]}(\zeta, \alpha)(X_{[1,m]}) = \mathrm{Tr}_{a,A}(Y_{a,c,A}\mathbb{T}_{\{a,A\},[1,m]}(\zeta, \alpha)\zeta^{\alpha - \mathbb{S}_{[1,m]}}(q^{-2S_{[1,m]}} X_{[1,m]})),$$
$$Y_{a,c,A} = -\tfrac{1}{2}\sigma_c^3\sigma_a^+ + \sigma_c^+\sigma_a^3 - \mathbf{a}_A\sigma_c^+\sigma_a^3.$$

For the proof of lemma 6.1 we compare the singularities of $\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})$ inside the functional $Z^{\kappa}_{[1,m+1]}$ with those of $\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})$ inside $Z^{\kappa}_{[1,m]}$.

It is known that $\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})$ is regular at $\zeta^2 = \xi_j^2$. Now we compare $\mathrm{res}_{\zeta^2=q^{\pm 2}\xi_m^2}\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})$ with $\mathrm{res}_{\zeta^2=\xi_m^2}\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})$. Set

$$U_{[1,m]} = \mathrm{res}_{\zeta^2=\xi_m^2}\mathbf{q}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})\frac{d\zeta^2}{\zeta^2},$$
$$\mathbf{q}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) = \mathrm{Tr}_A\left(\mathbb{T}_{A,[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}(q^{-2S_{[1,m]}}X_{[1,m]})\right).$$

Lemma A.1. *The operator* $\left[\sigma_m^+, U_{[1,m]}\right]_+$ *has its support in* $[1, m-1]$:

$$\left[\sigma_m^+, U_{[1,m]}\right]_+ = x_{[1,m-1]}I_m, \qquad x_{[1,m-1]} = \mathrm{Tr}_m\left(\sigma_m^+ U_{[1,m]}\right). \tag{A.6}$$

We have

$$\mathrm{res}_{\zeta^2=\xi_m^2}\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})\frac{d\zeta^2}{\zeta^2} = -\frac{1}{2}\left[\sigma_m^+, U_{[1,m]}\right]_+, \tag{A.7}$$

$$\mathrm{res}_{\zeta^2=q^{2\epsilon}\xi_m^2}\left(\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]}) + \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})\right)\frac{d\zeta^2}{\zeta^2}$$
$$= \begin{cases} -\frac{1}{4}\left[\sigma_m^+, U_{[1,m]}\right]_+ - U_{[1,m]}\left(\tau_m^+\sigma_c^+ - \sigma_m^+\tau_c^+\right) & (\epsilon = +); \\ \frac{1}{4}\left[\sigma_m^+, U_{[1,m]}\right]_+ + \left(\tau_m^-\sigma_c^+ - \sigma_m^+\tau_c^-\right)U_{[1,m]} & (\epsilon = -). \end{cases} \tag{A.8}$$

Proof. Property (A.6) appears in [3] as lemma 2.6 (see also [2], appendix D). Formula (A.7) is proved in [2], lemma 2.2. The calculation for (A.8) is similar, but we omit the details. □

Corollary A.2. *We have the relations between* $\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)$ *and* $\mathbf{c}_{[1,m]}(\zeta,\alpha)$

$$\mathrm{res}_{\zeta^2=\xi_j^2}\left(\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha) + \mathbf{t}^*_{[1,m]}(\zeta,\alpha)\mathbf{c}_{[1,m]}(\zeta,\alpha)\right)(X_{[1,m]})\frac{d\zeta^2}{\zeta^2} = 0. \tag{A.9}$$

Proof. To see (A.9), it suffices to write

$$\mathrm{res}_{\zeta^2=\xi_m^2}\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) = \mathrm{Tr}_a\left(\sigma_a^+\mathbb{P}_{a,m}\mathbb{T}_{a,[1,m-1]}(\xi_m,\alpha)U_{[1,m]}\right)$$
$$= \mathrm{Tr}_a\left(\mathbb{P}_{a,m}\mathbb{T}_{a,[1,m-1]}(\xi_m,\alpha)\left((0,1)_m U_{[1,m]}\begin{pmatrix}1\\0\end{pmatrix}_m\right)\right),$$

and use (A.6) and (A.7) and R-matrix symmetry. □

Using lemma A.1, we obtain

$$\mathrm{res}_{\xi_{m+1}^2=q^2\xi_m^2} Z^{\kappa}_{[1,m+1]}\{2\mathbf{g}_{m+1,[1,m]}(\xi_{m+1},\alpha)(X_{[1,m]})\}\frac{d\xi_{m+1}^2}{\xi_{m+1}^2}$$
$$= \mathrm{res}_{\zeta^2=\xi_m^2} Z^{\kappa}_{[1,m]}\{\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})\}\frac{d\zeta^2}{\zeta^2}$$
$$- 2\,\mathrm{res}_{\xi_{m+1}^2=q^2\xi_m^2} Z^{\kappa}_{[1,m+1]}\left\{U_{[1,m]}\left(\tau_m^+\sigma_{m+1}^+ - \sigma_m^+\tau_{m+1}^+\right)\right\}. \tag{A.10}$$

Here the third term of (A.5) does not contribute, because the only singularities of $\mathbf{f}_{[1,m]}(\zeta,\alpha)$ are the simple poles at $\zeta^2 = \xi_j^2$, and the following inhomogeneous analogue of theorem 3.1 holds

$$Z^{\kappa}_{[1,m+1]}\{\mathbb{T}_{m+1,[1,m]}(\xi_{m+1},\alpha)X_{[1,m]}\} = 2\rho(\xi_{m+1})Z^{\kappa}_{[1,m]}\{X_{[1,m]}\}. \tag{A.11}$$

Note that

$$\tau_m^+\sigma_{m+1}^+ - \sigma_m^+\tau_{m+1}^+ = \left(\tau_m^+\sigma_{m+1}^+ - \sigma_m^+\tau_{m+1}^+\right)P^-_{m,m+1},$$

where P^- is the projector on the singlet,

$$P^-(v_\varepsilon \otimes v_{\varepsilon'}) = \epsilon\delta_{\epsilon+\epsilon',0}\tfrac{1}{2}(v_+ \otimes v_- - v_- \otimes v_+), \qquad \sigma^3 v_\varepsilon = \varepsilon v_\varepsilon.$$

Using the cyclicity of trace and the quantum determinant relation

$$P^-_{m,m+1}T_{m,\mathbf{M}}(\xi_m)T_{m+1,\mathbf{M}}(q\xi_m) = a(q\xi_m)d(\xi_m)P^-_{m,m+1},$$

we find

$$\begin{aligned}
\mathrm{res}_{\xi^2_{m+1}=q^2\xi^2_m} Z^\kappa_{[1,m+1]}\big\{U_{[1,m]}\big(\tau^+_m\sigma^+_{m+1} - \sigma^+_m\tau^+_{m+1}\big)\big\} &= \frac{a(q\xi_m)d(\xi_m)}{\prod_{j=1}^m T(\xi_j,\kappa)\cdot T(q\xi_m,\kappa)}\\
&\quad\times \frac{\mathrm{Tr}_{[1,m+1]}\langle\kappa+\alpha|T_{[1,m-1],\mathbf{M}}(\boldsymbol{\xi},\kappa)U_{[1,m]}\big(\tau^+_m\sigma^+_{m+1} - \sigma^+_m\tau^+_{m+1}\big)|\kappa\rangle}{\langle\kappa+\alpha|\kappa\rangle}\\
&= -\frac{a(q\xi_m)d(\xi_m)}{T(\xi_m,\kappa)T(q\xi_m,\kappa)} Z^\kappa_{[1,m-1]}\left\{\mathrm{Tr}_m\frac{1}{2}\big[\sigma^+_m, U_{[1,m]}\big]_+\right\}\\
&= \frac{2a(q\xi_m)d(\xi_m)}{T(\xi_m,\kappa)T(q\xi_m,\kappa)}\,\mathrm{res}_{\zeta^2=\xi^2_m} Z^\kappa_{[1,m]}\{\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})\}\frac{\mathrm{d}\zeta^2}{\zeta^2}.
\end{aligned}$$

In the last line we used (A.6) and (A.7). Computation of the residue at $\zeta^2 = q^{-2}\xi^2_m$ is done similarly, using

$$\tau^-_m\sigma^+_{m+1} - \sigma^+_m\tau^-_{m+1} = P^-_{m,m+1}\big(\tau^-_m\sigma^+_{m+1} - \sigma^+_m\tau^-_{m+1}\big).$$

The residues at $\xi^2_{m+1} = q^{\pm 2}\xi^2_j$ are readily found from the *R*-matrix symmetry.

Lemma A.3.

$$\begin{aligned}
&\mathrm{res}_{\xi^2_{m+1}=q^{\pm2}\xi^2_j} Z^\kappa_{[1,m+1]}\{2\mathbf{g}_{m+1,[1,m]}(\xi_{m+1},\alpha)(X_{[1,m]})\}\frac{\mathrm{d}\xi^2_{m+1}}{\xi^2_{m+1}}\\
&\quad= \mathrm{res}_{\zeta^2=q^{\pm2}\xi^2_j}\omega(\zeta,\xi_j)\,\mathrm{res}_{\zeta^2=\xi^2_j} Z^\kappa_{[1,m]}\{\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})\}\frac{\mathrm{d}\zeta^2}{\zeta^2}. \qquad (A.12)
\end{aligned}$$

Proof. This follows from the preceding calculations and

$$\mathrm{res}_{\zeta^2=q^2\xi^2_j}\omega(\zeta,\xi_j) = 1 - \frac{4a(\xi_j)d(q\xi_j)}{T(\xi_j,\kappa)T(\xi_j q,\kappa)},$$

$$\mathrm{res}_{\zeta^2=q^{-2}\xi^2_j}\omega(\zeta,\xi_j) = -\left(1 - \frac{4a(\xi_j q^{-1})d(\xi_j)}{T(\xi_j,\kappa)T(\xi_j q^{-1},\kappa)}\right).$$

□

Let us return to the homogeneous case $\xi_1 = \cdots = \xi_m = 1$. The operators $\mathbf{c}(\zeta,\alpha)$, $\bar{\mathbf{c}}(\zeta,\alpha)$ acting from $\mathcal{W}_{\alpha+1,-1}$ to $\mathcal{W}_{\alpha,0}$ are defined by

$$\begin{aligned}
\mathbf{c}(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]}) &= q^{2\alpha S(0)}\mathbf{c}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}),\\
\bar{\mathbf{c}}(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]}) &= q^{2\alpha S(0)}\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})
\end{aligned} \qquad (A.13)$$

and the requirement of translational invariance. This definition is equivalent to that given in [2]. The reduction relation proven there ensures the self-consistency of the present definition. Equation (6.6) follows by writing (A.12) as a contour integral and specializing to $\xi_1 = \cdots = \xi_m = 1$.

It remains to show that the polynomial $P_{\mathbf{n}}(\zeta^2)$ in the remainder term of (6.6) has degree at most $\mathbf{n}$. The only nontrivial case to consider is when the spin of $X_{[1,m]}$ equals -1. Then it follows from the fact [2] that

$$\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]}) = O(1), \qquad \zeta^2 \to \infty.$$

This finishes the proof of lemma 6.1.

Appendix B. Proof of lemma 6.2

The goal of this section is to prove lemma 6.2. The proof is done in several steps.

Step 1. Recall definition (A.3). Fix a solution $\mathbf{f}_{0,[1,m]}(\zeta,\alpha)$ of the equation

$$\Delta_\zeta \mathbf{f}_{0,[1,m]}(\zeta,\alpha)(X_{[1,m]}) = \mathbf{k}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) \tag{B.1}$$

which has poles only at $\zeta^2 = q^{2n}\,(n\in\mathbb{Z})$. Define further

$$\mathbf{b}_0^*(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]}) = \lim_{l\to\infty} q^{2\alpha S(0)}\,\mathrm{Tr}_c\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{g}_{0,c,[1,m]}(\zeta,\alpha)(X_{[1,m]})\}, \tag{B.2}$$

$$\begin{aligned}\mathbf{g}_{0,c,[1,m]}(\zeta,\alpha)(X_{[1,m]}) &= \tfrac{1}{2}\mathbf{f}_{0,[1,m]}(q\zeta,\alpha)(X_{[1,m]}) + \tfrac{1}{2}\mathbf{f}_{0,[1,m]}(q^{-1}\zeta,\alpha)(X_{[1,m]})\\ &\quad - \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{0,[1,m]}(\zeta,\alpha)(X_{[1,m]}) + \mathbf{u}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]}).\end{aligned} \tag{B.3}$$

Lemma B.1. *Define*

$$\begin{aligned}I_{\mathbf{M}}(\zeta)(X_{[1,m]}) &= Q^-_{\mathbf{M}}(\zeta,\kappa+\alpha)\,\mathrm{Tr}_{[1,m],c}(T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}\\ &\quad\times(\zeta,\kappa)\mathbf{g}_{0,c,[1,m]}(\zeta,\alpha)(X_{[1,m]}))\,Q^+_{\mathbf{M}}(\zeta,\kappa).\end{aligned}$$

Identity (6.7) follows from

$$\int_{\Gamma_{\mathbf{m}}} I_{\mathbf{M}}(\zeta)(X_{[1,m]})\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = 0. \tag{B.4}$$

Proof. Introduce the operator

$$D_\zeta F(\zeta) = F(q\zeta) + F(q^{-1}\zeta) - \mathbf{t}^*(\zeta)F(\zeta),$$

which can be used to rewrite the definition of $\mathbf{b}^*(\zeta,\alpha)$:

$$\begin{aligned}\mathbf{b}^*(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]}) &= D_\zeta(q^{2\alpha S(0)}\mathbf{f}^*_{[1,m]}(\zeta,\alpha)(X_{[1,m]}))\\ &\quad + q^{2\alpha S(0)}\lim_{l\to\infty}\mathrm{Tr}_c\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{u}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})\}.\end{aligned}$$

In similar formula for $\mathbf{b}_0^*(\zeta,\alpha)$ the only change is $\mathbf{f}$ to $\mathbf{f}_0$, $\mathbf{u}$ remains the same. Comparing this with equations (A.4), (B.1) and (A.13) we arrive at

$$\mathbf{b}_0^*(\zeta,\alpha) - \mathbf{b}^*(\zeta,\alpha) = D_\zeta\Delta_\zeta^{-1}(\mathbf{c}(\zeta q,\alpha) + \mathbf{c}(\zeta q^{-1},\alpha) + \bar{\mathbf{c}}(\zeta,\alpha)).$$

Now consider the term containing $\mathbf{c}$ in (6.7). Note that due to lemma 3.1 for any quasi-local operator $X(\zeta)$ depending on ζ

$$Z^\kappa\{\overline{D}_\zeta(X(\zeta))\} = Z^\kappa\{D_\zeta(X(\zeta))\}.$$

So we replace in (6.7) $\overline{D}_\zeta$, $\overline{D}_\xi$ by D_ζ, D_ξ. On the other hand, from (A.9) it follows that we have an equality of formal power series in $(\zeta^2-1)^{-1}$,

$$\begin{aligned}&\bar{\mathbf{c}}(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]})\\ &\quad = -\frac{1}{2\pi\mathrm{i}}\oint_\Gamma \psi(\zeta/\xi,\alpha)\mathbf{t}^*(\xi,\alpha)\mathbf{c}(\xi,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]})\frac{\mathrm{d}\xi^2}{\xi^2}.\end{aligned} \tag{B.5}$$

Using (B.5) we evaluate

$$\mathbf{c}(\zeta q,\alpha) + \mathbf{c}(\zeta q^{-1},\alpha) + \bar{\mathbf{c}}(\zeta,\alpha) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma D_\xi(\psi(\zeta/\xi,\alpha))\mathbf{c}(\xi,\alpha)\frac{\mathrm{d}\xi^2}{\xi^2}.$$

In other words we obtain an equation similar to (6.4) with $\mathbf{b}^*$ replaced by the expression under Z^κ in (6.7), and $\mathbf{g}$ replaced by $\mathbf{g}_0$, which is nothing but the matrix element of (B.4). □

Step 2. The next step is to reduce the identity to a difference equation for $\mathbf{g}_0$ on a finite interval. We will show that, for all $\mathbf{m} = \mathbf{1}, \ldots, \mathbf{n}$, identity (B.4) reduces to the same equation for a quantity in the space direction. So, we can forget the Matsubara direction. Introduce an operator

$$\mathbb{A}_{c,[1,m]}(\zeta)(Y_{[1,m]\sqcup c}) = T_{c,[1,m]}(\zeta)q^{\alpha\sigma_c^3}\theta_c\left(Y_{[1,m]\sqcup c}\theta_c(T_{c,[1,m]}(\zeta)^{-1})\right),$$

where θ signifies the anti-involution

$$\theta(x) = \sigma^2 x^t \sigma^2 \qquad (x \in \mathrm{End}(V)).$$

Lemma B.2. *Identity (B.4) follows from the equation*

$$\mathbf{g}_{0,c,[1,m]}(\zeta,\alpha)(X_{[1,m]}) = -\mathbb{A}_{c,[1,m]}(\zeta)(\mathbf{g}_{0,c,[1,m]}(q^{-1}\zeta,\alpha)(X_{[1,m]})). \tag{B.6}$$

Proof. By symmetry it suffices to consider the case $\mathbf{m} = \mathbf{n}$. We prove the assertion assuming that $\mathbf{s_n} = 1/2$. The general case is reduced to this case by the standard fusion procedure.

From the defining relation (5.1) for $\varphi(\zeta)$, we have

$$\mathrm{res}_{\zeta^2=q^{-2}\tau_{\mathbf{n}}^2}\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} = \frac{a(\tau_{\mathbf{n}})}{d(q^{-1}\tau_{\mathbf{n}})}\,\mathrm{res}_{\zeta^2=\tau_{\mathbf{n}}^2}\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}.$$

So, equation (B.4) is equivalent to

$$d(\tau_{\mathbf{n}}q^{-1})I_{\mathbf{M}}(\tau_{\mathbf{n}})(X_{[1,m]}) + a(\tau_{\mathbf{n}})I_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1})(X_{[1,m]}) = 0. \tag{B.7}$$

Let us compute $I_{\mathbf{M}}(\tau_{\mathbf{n}})(X_{[1,m]})$. We simplify notations introducing

$$Y_{[1,m]}(\zeta,\alpha) = \mathbf{g}_{0,c,[1,m]}(\zeta,\alpha)(X_{[1,m]}).$$

First, move $T_{c,\mathbf{M}}(\tau_{\mathbf{n}})$ to the left using the Yang–Baxter equation:

$$\begin{aligned}&\mathrm{Tr}_{[1,m],c}(T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\tau_{\mathbf{n}},\kappa)Y_{[1,m]}(\tau_{\mathbf{n}},\alpha))\\ &\quad = \mathrm{Tr}_{[1,m],c}(T_{c,\mathbf{M}}(\tau_{\mathbf{n}},\kappa)T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,[1,m]}(\tau_{\mathbf{n}})^{-1}Y_{[1,m]}(\tau_{\mathbf{n}},\alpha)T_{c,[1,m]}(\tau_{\mathbf{n}})).\end{aligned}$$

Now, for $s_{\mathbf{n}} = 1/2$, the L operator satisfies

$$L_{c,\mathbf{n}}(1) = \eta P_{c,\mathbf{n}},$$

where we have set $\eta = q^{1/2}(q-q^{-1})$. Note that

$$\begin{aligned}&T_{c,\mathbf{M}}(\tau_{\mathbf{n}},\kappa) = \eta P_{c,\mathbf{n}}T_{c,\mathbf{M}'}(\tau_{\mathbf{n}},\kappa) = \eta T_{\mathbf{n},\mathbf{M}'}(\tau_{\mathbf{n}},\kappa)P_{c,\mathbf{n}},\\ &\eta T_{\mathbf{n},\mathbf{M}'}(\tau_{\mathbf{n}},\kappa) = T_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa+\alpha)q^{-\alpha\sigma_{\mathbf{n}}^3},\end{aligned}$$

where $\mathbf{M}'$ signifies the subinterval $[\mathbf{1},\mathbf{n}-\mathbf{1}]$. Moreover,

$$T_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa+\alpha)Q_{\mathbf{M}}^-(\tau_{\mathbf{n}},\kappa+\alpha) = a(\tau_{\mathbf{n}})Q_{\mathbf{M}}^-(\tau_{\mathbf{n}}q^{-1},\kappa+\alpha),$$

because $d(\tau_{\mathbf{n}}) = 0$. So we can evaluate $I_{\mathbf{M}}(\tau_{\mathbf{n}})(X_{[1,m]})$ as

$$\begin{aligned}I_{\mathbf{M}}(\tau_{\mathbf{n}})(X_{[1,m]}) &= a(\tau_{\mathbf{n}})Q_{\mathbf{M}}^-(\tau_{\mathbf{n}}q^{-1},\kappa+\alpha)\mathrm{Tr}_{[1,m],c}(P_{c,\mathbf{n}}T_{[1,m],\mathbf{M}}(1,\kappa)\\ &\quad\times q^{-\alpha\sigma_c^3}T_{c,[1,m]}(\tau_{\mathbf{n}})^{-1}Y_{[1,m]}(\tau_{\mathbf{n}},\alpha)T_{c,[1,m]}(\tau_{\mathbf{n}}))Q_{\mathbf{M}}^+(\tau_{\mathbf{n}},\kappa).\end{aligned}$$

Now note that

$$T_{[1,m],\mathbf{M}}(1,\kappa) = T_{[1,m],\mathbf{n}}(\tau_{\mathbf{n}}^{-1})T_{[1,m],\mathbf{M}'}(1,\kappa) = \mu(\tau_{\mathbf{n}})T_{\mathbf{n},[1,m]}(\tau_{\mathbf{n}})^{-1}T_{[1,m],\mathbf{M}'}(1,\kappa),$$

where $\mu(\tau)$ is a function whose explicit form is irrelevant for our calculation.

Bring the permutation through $T_{[1,m],\mathbf{M}}$ and put $T_{[1,m],c}$ to the right by cyclicity of trace:

$$I_{\mathbf{M}}(\tau_{\mathbf{n}})(X_{[1,m]}) = \mu(\tau_{\mathbf{n}})a(\tau_{\mathbf{n}})Q^{-}_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1}, \kappa+\alpha) \times \mathrm{Tr}_{[1,m],c}(T_{[1,m],\mathbf{M}'}(1,\kappa)P_{c,\mathbf{n}}q^{-\alpha\sigma_c^3}T_{c,[1,m]}(\tau_{\mathbf{n}})^{-1}Y_{[1,m]}(\tau_{\mathbf{n}},\alpha))Q^{+}_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa).$$

We compute $I_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1})(X_{[1,m]})$ similarly, using

$$T_{c,\mathbf{M}}(\tau_{\mathbf{n}}q^{-1},\kappa) = -2\eta q^{-1}P^{-}_{c,\mathbf{n}}T_{c,\mathbf{M}'}(\tau_{\mathbf{n}}q^{-1},\kappa) = 2P^{-}_{c,\mathbf{n}}T_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1}\kappa),$$
$$T_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1},\kappa)Q^{+}_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1},\kappa) = d(\tau_{\mathbf{n}}q^{-1})Q^{+}_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa).$$

The result is

$$I_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1})(X_{[1,m]}) = \mu(\tau_{\mathbf{n}})d(\tau_{\mathbf{m}}q^{-1})Q^{-}_{\mathbf{M}}(\tau_{\mathbf{n}}q^{-1},\kappa+\alpha) \times \mathrm{Tr}_{[1,m],c}(T_{[1,m],\mathbf{M}'}(1,\kappa)P_{c,\mathbf{n}}\theta_c(Y_{[1,m]}(\tau_{\mathbf{n}}q^{-1},\alpha)\theta_c(T_{c,[1,m]}(\tau_{\mathbf{n}})^{-1})))Q^{+}_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa),$$

where we applied the anti-automorphism θ_c under Tr_c using $\mathrm{Tr}(\theta(x)) = \mathrm{Tr}(x)$ and $\theta_c(2P^{-}_{c,\mathbf{n}}) = P_{c,\mathbf{n}}$. Thus (B.6) implies (B.7). □

Step 3. The third step is to reduce (B.6) to representation theory.

Lemma B.3. *Set*

$$y_1 = \tau_c^{-}\sigma_a^{+} - \tau_a^{-}\sigma_c^{+},$$
$$y_2 = \sigma_c^{-}\sigma_a^{+} + \tau_c^{-}\tau_a^{+} - \tau_c^{+}\tau_a^{-} + \left(\tau_c^{-}\sigma_a^{+} - \sigma_c^{+}\tau_a^{+}\right)\mathbf{a}_A - \sigma_c^{+}\sigma_a^{+}\mathbf{a}_A^2.$$

Then equation (B.6) is equivalent to the identities

$$\mathrm{Tr}_{c,a,A}(y\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{a,[1,m]}(q^{-1}\zeta,\alpha)\mathbb{T}_{A,[1,m]}(q^{-1}\zeta,\alpha)(X'_{[1,m]})) = 0 \quad (y = y_1, y_2), \tag{B.8}$$

where we have set $X'_{[1,m]} = (q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}q^{-2S_{[1,m]}}X_{[1,m]}$.

Proof. Recall definitions (B.1) and (B.3). From the definition of $\mathbb{A}_{c,[1,m]}$ we easily find that

$$\mathbb{A}_{c,[1,m]}(\zeta)(x_cX_{[1,m]}) = \mathbb{T}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})\theta_c(x_c), \tag{B.9}$$

$$\mathbb{A}_{c,[1,m]}(\zeta)\mathbb{T}_{c,[1,m]}(q^{-1}\zeta,\alpha)(X_{[1,m]}) = X_{[1,m]}. \tag{B.10}$$

Write

$$\mathbf{u}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]}) = -\tfrac{1}{2}\sigma_c^3\mathbf{k}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) + \sigma_c^{+}\mathbf{l}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}).$$

Applying (B.9) and (B.10) we reduce (B.6) to the form

$$0 = (\mathbf{k}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) - \mathbb{T}_{c,[1,m]}(\zeta,\alpha)(\mathbf{k}_{[1,m]}(q^{-1}\zeta,\alpha)(X_{[1,m]})))\tau_c^{-} + (\mathbf{l}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) - \mathbb{T}_{c,[1,m]}(\zeta,\alpha)(\mathbf{l}_{[1,m]}(q^{-1}\zeta,\alpha)(X_{[1,m]})))\sigma_c^{+}.$$

We rewrite this further, using the fact that $y_c = 0$ if and only if $\mathrm{Tr}_c(x_cy_c) = 0$ for any $x_c \in \mathrm{End}(V_c)$. Nontrivial conditions arise from the choices $x_c = \tau_c^{-}$ or σ_c^{-}, giving respectively

$$\mathbf{k}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) = \mathrm{Tr}_{c,a,A}\big(\big(\sigma_a^{+}\tau_c^{-} + \big(\sigma_a^3 - \mathbf{a}_A\sigma_a^{+}\big)\sigma_c^{+}\big) \times \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}(q^{-2S_{[1,m]}}X_{[1,m]})\big), \tag{B.11}$$

$$\mathbf{l}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) = \mathrm{Tr}_{c,a,A}\big(\big(\sigma_a^{+}\sigma_c^{-} + \big(\sigma_a^3 - \mathbf{a}_A\sigma_a^{+}\big)\tau_c^{+}\big) \times \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}(q^{-2S_{[1,m]}}X_{[1,m]})\big). \tag{B.12}$$

On the other hand, we have an identity (see [2], (2.20))

$$\mathbb{T}_{A,[1,m]}(\zeta,\alpha)(X_{[1,m]}) = \mathrm{Tr}_a\big(\tau_a^{+}\mathbb{T}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}}(X_{[1,m]})\big),$$

which allows us to rewrite the left-hand side of (B.11) as

$$\mathbf{k}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})$$
$$= \mathrm{Tr}_{c,a,A}\left(\sigma_c^+\tau_a^+\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right).$$

For $\mathbf{l}_{[1,m]}(\zeta,\alpha)(X_{[1,m]})$, we obtain an analogous expression, replacing $\sigma_c^+\tau_a^+$ by $\left(\sigma_c^3+\mathbf{a}_A\sigma_c^+\right)\tau_a^+$.

After this rewriting we take the difference of the left- and the right-hand sides of (B.11) and (B.12), and do further the gauge transformation

$$F_{a,A}\cdot\mathbb{T}_{\{a,A\},[1,m]}(\zeta,\alpha)\cdot F_{a,A}^{-1} = \mathbb{T}_{a,[1,m]}(\zeta,\alpha)\mathbb{T}_{A,[1,m]}(\zeta,\alpha) \qquad \left(F_{a,A} = 1-\mathbf{a}_A\sigma_a^+\right).$$

The assertion of lemma follows. □

To finish, it remains to prove (B.8). Let $\mathcal{R}$ be the universal R matrix of $U_q(\widehat{\mathfrak{sl}}_2)$. Denote by π_ζ the evaluation module over $V = \mathbb{C}^2$, and by ϖ_ζ that of q-oscillator representation W. For the notation and details, we refer to appendix A of [2]. Let further $\pi_{[1,m]} = \pi_1^{\otimes m}$. The identities of lemma B.3 can be written as

$$\mathrm{Tr}_{c,a,A}\left(y(\pi_\zeta\otimes\pi_{q^{-1}\zeta}\otimes\varpi_{q^{-1}\zeta}\otimes\pi_{[1,m]})\mathcal{R}\right) = 0 \qquad (y = y_1, y_2). \tag{B.13}$$

In the tensor product

$$Z = V_\zeta\otimes V_{q^{-1}\zeta}\otimes W_{q^{-1}\zeta},$$

there are two pairs which allow for nontrivial $U_q\mathbf{b}$-submodules:

$$V_0\subset V_\zeta\otimes V_{q^{-1}\zeta}, \qquad W_0\subset V_{q^{-1}\zeta}\otimes W_{q^{-1}\zeta}.$$

The submodule $V_0\simeq\mathbb{C}$ (resp. $W_0\simeq W_{q^{-2}\zeta}$) is spanned by $v_+\otimes v_- - v_-\otimes v_+$ (resp. $\{v_-\otimes|n\rangle+(q^{2n}-1)v_+\otimes|n-1\rangle\}_{n\geqslant 0}$). Set

$$Z_1 = V_0\otimes W_{q^{-1}\zeta}, \qquad Z_2 = V_\zeta\otimes W_0.$$

Then a direct calculation shows that

$$y_1 Z\subset Z_1, \qquad y_1 Z_1 = 0,$$
$$y_2 Z\subset Z_1\oplus Z_2, \qquad y_2 Z_1\subset Z_2, \qquad y_2 Z_2\subset Z_1.$$

Since $xZ_i\subset Z_i$ $(i=1,2)$ holds for $x\in U_q\mathbf{b}$, we have

$$\mathrm{Tr}_{c,a,A}\left(y_i(\pi_\zeta\otimes\pi_{q^{-1}\zeta}\otimes\varpi_{q^{-1}\zeta})(x)\right) = 0 \qquad (x\in U_q\mathbf{b}).$$

The proof is now complete.

Step 4. To complete the proof of lemma 6.2, we show the matrix element of (B.4) between $\langle\kappa+\alpha|, |\kappa\rangle$ for $\mathbf{m}=\mathbf{0}$. The integral consists of two parts, say J_1 and J_2, coming from the first three terms in (B.3), or from $\mathbf{u}_{c,[1,m]}(\zeta,\alpha)$, respectively. We note that in view of (B.1) and lemma 5.1 the proper meaning of J_1 is

$$J_1 = \frac{1}{2}\int_{\Gamma_0}\langle\kappa+\alpha|\mathrm{Tr}_{[1,m],c}\left(T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa)\mathbf{k}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})\right)|\kappa\rangle$$
$$\times Q^-(\zeta,\kappa+\alpha)\left(a(\zeta)Q^+(q^{-1}\zeta,\kappa)-d(\zeta)Q^+(q\zeta,\kappa)\right)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}.$$

The functions

$$T_{c,\mathbf{M}}(\zeta,\kappa),\ \varphi(\zeta),\ \zeta^{\kappa+\alpha}Q^-(\zeta,\kappa+\alpha),\ \zeta^{-\kappa}Q^+(\zeta,\kappa),\ a(\zeta),\ d(\zeta)$$

are all regular at $\zeta^2=0$. On the other hand, for $X_{[1,m]}$ of spin -1, we have the estimate

$$\zeta^{-\alpha}\mathbf{k}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) = O(\zeta^2) \qquad (\zeta^2\to 0),$$

as explained in [2], section 2.5. The same argument there shows also

$$\zeta^{-\alpha}\mathbf{l}_{[1,m]}(\zeta,\alpha)(X_{[1,m]}) = O(\zeta^2) \qquad (\zeta^2\to 0).$$

It follows that in both J_1 and J_2 the integrand is regular and the residue at $\zeta^2=0$ vanishes. This completes the proof of lemma 6.2.

Appendix C. Classical limit

In this appendix, we explain the classical limit of our construction and its relation to hyperelliptic Riemann surfaces. We shall not go into much detail since similar considerations were discussed in [10, 12]. We assume $\alpha = 0$. At the moment we are not ready to discuss the classical limit in the case $\alpha \neq 0$. We consider Bethe vectors of spin 0, so that $\zeta^{\mp\kappa} Q^{\pm}(\zeta, \kappa)$ are polynomials in ζ^2 of the same degree

$$s = \sum_{\mathbf{m}=1}^{\mathbf{n}} s_{\mathbf{m}}.$$

In the parametrization $q = \mathrm{e}^{\pi \mathrm{i}\nu}$, the classical limit amounts to $\nu \to 0$. So ν plays the role of Planck's constant. Let us see what happens to the solutions to the Baxter equation in this limit. First of all, in order to obtain a Riemann surface of a finite genus, we keep $\mathbf{n}$ finite. But for the classical limit we need to have large quantum numbers. This is achieved by considering large spins $s_{\mathbf{m}}$. Actually, this was the main reason for us to consider arbitrary spins in the Matsubara direction. So, we require that $\nu s_{\mathbf{m}}$, or equivalently $q^{s_{\mathbf{m}}}$, tend to fixed non-zero values when $\nu \to 0$. Similarly, we demand that $\nu\kappa$, or q^{κ}, stays finite in the limit. In this situation, $a(\zeta)$, $d(\zeta)$ and $T(\zeta) = T(\zeta, \kappa)$ tend to polynomials in ζ^2 of degree $\mathbf{n}$, which we denote by the same letters as in the quantum case.

In the classical limit, the poles of $\varphi(\zeta)$ concentrate on the portion of circles between the end points $\tau_{\mathbf{m}}^2 q^{-2s_{\mathbf{m}}}$ and $\tau_{\mathbf{m}}^2 q^{2s_{\mathbf{m}}}$. The s zeros of the polynomials $\zeta^{\mp\kappa} Q^{\pm}(\zeta, \kappa)$ concentrate to $\mathbf{n}$ open curved segments $C_{\mathbf{m}}$ close to the above circular segments. This claim is difficult to prove, but we can justify them by analysing the Baxter equation in the classical limit.

Consider the Baxter equation

$$d(\zeta)Q(\zeta q) + a(\zeta)Q(\zeta q^{-1}) = T(\zeta)Q(\zeta). \tag{C.1}$$

We look for its quasi-classical solution in the form

$$Q(\zeta) = F(\zeta, \nu) \exp\left\{\frac{1}{2\pi \mathrm{i}\nu} \int_1^{\zeta^2} \log \eta(\xi) \frac{\mathrm{d}\xi^2}{\xi^2}\right\}, \tag{C.2}$$

where $\eta(\zeta)$ is a function independent of ν and $F(\zeta, \nu)$ is a power series in ν. First, dividing the Baxter equation by $Q(\zeta)$ and considering the leading order in Planck's constant we conclude that $\eta(\zeta)$ must solve the equation

$$d(\zeta)\eta(\zeta) + a(\zeta)\eta^{-1}(\zeta) = T(\zeta). \tag{C.3}$$

This is the equation of the spectral curve of the corresponding classical model.

The function $\eta(\zeta)$ has two branches,

$$\eta^{\pm}(\zeta) = \frac{T(\zeta) \pm \sqrt{T(\zeta)^2 - 4a(\zeta)d(\zeta)}}{2d(\zeta)},$$

for future convenience we choose the branch of the square root such that $\sqrt{(q^{\kappa} - q^{-\kappa})^2} = q^{\kappa} - q^{-\kappa}$.

Consider the behaviour $\zeta^2 \to \infty$. The polynomial $T(\zeta)$ is not arbitrary, it comes from the quasi-classical limit of a solution to the Baxter equation (C.1). Recall that for large ζ the function $Q^{\pm}(\zeta)$ is $O(\zeta^{\pm\kappa+2s})$ as discussed in section 4. Also we have for $\zeta^2 \to \infty$ and to the main order in Planck's constant

$$a(\zeta) = \tau^{-2} q^{2s} \zeta^{2\mathbf{n}} + \cdots, \qquad \mathrm{d}(\zeta) = \tau^{-2} q^{-2s} \zeta^{2\mathbf{n}} + \cdots,$$

where $\tau = \prod \tau_{\mathbf{m}}$. So, the Baxter equation implies that

$$T(\zeta) = \tau^{-2}(q^{\kappa} + q^{-\kappa})\zeta^{2\mathbf{n}} + \cdots.$$

Hence when $\zeta^2 \to \infty$ we have $\eta^{\pm} \to q^{\pm\kappa+2s}$, which means that η^+ (resp. η^-) corresponds to quasi-classical limit of Q^+ (resp. Q^-).

Throughout this paper we use as parameter ζ while all the important quantities are actually functions of ζ^2. This notational problem is due to historical reasons, and we are forced to tolerate it in the quantum case. However, in the classical case this notation becomes very unnatural making incomprehensible simple formulae for differentials on hyperelliptic Riemann surface. That is why in what follows we shall often use the parameter $z = \zeta^2$. For the same reason we denote the discriminant $T(\zeta)^2 - 4a(\zeta)d(\zeta)$, which actually depends on ζ^2, by $P(\zeta^2)$. Recalling that $a(\zeta), d(\zeta), T(\zeta)$ are polynomials of ζ^2 and making the change of variables, $z = \zeta^2$, $w = 2d(\zeta)\eta(\zeta) - T(\zeta)$, we bring the spectral curve (C.3) to the canonical form:

$$w^2 = P(z). \tag{C.4}$$

In the q-deformed Abelian integrals the integration measure contains $Q^-(\zeta,\kappa)Q^+(\zeta,\kappa)$. The most direct way to compute this quantity uses the quantum Wronskian (4.4)

$$\begin{aligned}\frac{1}{q^{\kappa}-q^{-\kappa}}W(\zeta) &= Q^+(\zeta,\kappa)Q^-(\zeta q,\kappa) - Q^-(\zeta,\kappa)Q^+(\zeta q,\kappa)\\ &\underset{\nu\to 0}{\to} (\eta^- - \eta^+)Q^+(\zeta,\kappa)Q^-(\zeta,\kappa).\end{aligned}$$

This implies

$$Q^+(\zeta,\kappa)Q^-(\zeta,\kappa)\varphi(\zeta) \underset{\nu\to 0}{\to} \frac{1}{q^{-\kappa}-q^{\kappa}}\frac{1}{\sqrt{P(\zeta^2)}}, \tag{C.5}$$

where we used the identity $W(\zeta)d(\zeta)\varphi(\zeta) = 1$.

The discriminant $P(z)$ is a polynomial of degree $2\mathbf{n}$. Let us call its zeros $x_{\mathbf{1}},\ldots,x_{2\mathbf{n}}$. The Riemann surface (C.4) is presented as two copies of the z-plane glued together along the cuts $[x_{2\mathbf{m}-\mathbf{1}}, x_{2\mathbf{m}}]$. According to the conjecture accepted previously, the branch points can be ordered in such a way that the cut $[x_{2\mathbf{m}-\mathbf{1}}, x_{2\mathbf{m}}]$ is not far from the location of poles of $\varphi(\zeta)$ which are contained in the contour $\Gamma_{\mathbf{m}}$. According to (C.5), for any polynomial $L(\zeta^2)$ we have in the classical limit

$$\int_{\Gamma_{\mathbf{m}}} L(\zeta^2)Q^+(\zeta,\kappa)Q^-(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2} \underset{\nu\to 0}{\to} \frac{2}{q^{-\kappa}-q^{\kappa}}\int_{c_{\mathbf{m}}}\frac{L(z)}{\sqrt{P(z)}}\frac{\mathrm{d}z}{z}, \tag{C.6}$$

where $c_{\mathbf{m}}$ is a contour going in the z-plane around $[x_{2\mathbf{m}-1}, x_{2\mathbf{m}}]$ for $\mathbf{1} \leqslant \mathbf{j} \leqslant \mathbf{n}$, or around 0 for $\mathbf{m} = \mathbf{0}$. The limit (C.6) requires some remarks. The integral on the left-hand side is taken over the contour $\Gamma_{\mathbf{m}}$. In the limit the integrand develops cuts which appear as a result of concentration of zeros of $Q^+(\zeta,\kappa)Q^-(\zeta,\kappa)$ and poles of $\varphi(\zeta)$. So, obviously, in the limiting process we have to deform the contour in order that it does not cross the cut. This is how the integral around $c_{\mathbf{m}}$ appears.

The Riemann surface (C.4) has genus $\mathbf{n}-\mathbf{1}$. The contours $c_{\mathbf{m}}$, with $\mathbf{m} = \mathbf{1},\ldots,\mathbf{n}-\mathbf{1}$ can be taken as a-cycles. Our Riemann surface has two points $0^{\pm}$ which lie on different sheets and project to $z = 0$. The contour $c_{\mathbf{0}}$ goes around 0^+. Similarly we have two points $\infty^{\pm}$ which project to $z = \infty$.

Define the differentials on the Riemann surface

$$\sigma_{\mathbf{j}}(z) = \frac{z^{\mathbf{j}-1}}{\sqrt{P(z)}}\mathrm{d}z, \qquad \mathbf{j} = \mathbf{0},\ldots,\mathbf{n}.$$

The differentials $\sigma_{\mathbf{j}}(z)$ where $\mathbf{j} = \mathbf{1},\ldots,\mathbf{n}-\mathbf{1}$, are holomorphic (the first kind) differentials, while the differentials $\sigma_{\mathbf{0}}$ and $\sigma_{\mathbf{n}}$ are the third kind differentials. The differential $\sigma_{\mathbf{0}}$ has simple poles at $z = 0^{\pm}$, it is dual to the contour $c_{\mathbf{0}}$. The differential $\sigma_{\mathbf{n}}$ has simple poles at $z = \infty^{\pm}$.

The holomorphic differentials can be normalized with respect to $c_{\mathbf{i}}, \mathbf{i} = \mathbf{1}, \ldots, \mathbf{n-1}$ because

$$\det\left(\int_{c_{\mathbf{i}}} \sigma_{\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}=\mathbf{1},\ldots,\mathbf{n-1}} \neq 0.$$

This is the classical version of (7.1).

Consider the differentials whose only singularities are at $\infty^{\pm}$. Among those are exact forms

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^k\sqrt{P(z)})\,\mathrm{d}z, \qquad z^k\,\mathrm{d}z, \qquad k \geqslant 0. \tag{C.7}$$

Up to exact forms, holomorphic forms and the third kind differential $\sigma_{\mathbf{n}}(z)$ there are $\mathbf{n-1}$ linearly independent second kind differentials with singularities at $\infty^{\pm}$:

$$\tilde{\sigma}_{\mathbf{j}}(z) = z^{\mathbf{j}}\left[\frac{\mathrm{d}}{\mathrm{d}z}(z^{-2\mathbf{j}}P(z))\right]_+ \frac{\mathrm{d}z}{2\sqrt{P(z)}}, \qquad \mathbf{j} = \mathbf{1}, \ldots, \mathbf{n-1},$$

where $[f(z)]_+$ means the polynomial part of $f(z)$, which is a Laurent polynomial at $z = \infty$. We shall use at some point the differential $\tilde{\sigma}_{\mathbf{0}}$ which is an exact form.

The most important identity in the theory of Riemann surfaces is the Riemann bilinear relations. Usually this identity is written in the form

$$\sum_{\mathbf{m}=\mathbf{1}}^{\mathbf{g}}\left(\int_{a_{\mathbf{m}}}\omega_1\int_{b_{\mathbf{m}}}\omega_2 - \int_{b_{\mathbf{m}}}\omega_1\int_{a_{\mathbf{m}}}\omega_2\right) = 2\pi\mathrm{i}\,\omega_1\circ\omega_2,$$

where $\omega_{1,2}$ are the first or the second kind differentials, and

$$\omega_1\circ\omega_2 = -\sum \mathrm{res}(\omega_1 d^{-1}\omega_2).$$

In our case the a-cycles coincide with $c_{\mathbf{1}}, \ldots, c_{\mathbf{n-1}}$. The b-cycle $b_{\mathbf{m}}$ $(\mathbf{m} = \mathbf{1}, \ldots, \mathbf{n-1})$ crosses the cycle $a_{\mathbf{m}}$ once on the first sheet of the surface, goes to the second sheet through the **m**th cut, arrives to **n**th cut by the second sheet, crosses this cut and returns by the first sheet to its beginning.

An alternative way of writing the Riemann bilinear relations is the following. It is easy to see that σ and $\tilde{\sigma}$ constitute a canonical basis

$$\sigma_{\mathbf{i}}\circ\tilde{\sigma}_{\mathbf{j}} = \delta_{\mathbf{i},\mathbf{j}}, \qquad \sigma_{\mathbf{i}}\circ\sigma_{\mathbf{j}} = 0, \qquad \tilde{\sigma}_{\mathbf{i}}\circ\tilde{\sigma}_{\mathbf{j}} = 0. \tag{C.8}$$

Now construct the antisymmetric form

$$\sigma(x,y) = \sum_{\mathbf{j}=\mathbf{1}}^{\mathbf{n-1}}(\sigma_{\mathbf{j}}(x)\tilde{\sigma}_{\mathbf{j}}(y) - \sigma_{\mathbf{j}}(y)\tilde{\sigma}_{\mathbf{j}}(x)). \tag{C.9}$$

Then

$$\int_{g_1}\int_{g_2}\sigma(x,y) = 2\pi\mathrm{i}\,g_{\mathbf{1}}\circ g_{\mathbf{2}}, \tag{C.10}$$

where on the right-hand side we put the intersection number of cycles. From the explicit formulae for $\sigma_{\mathbf{i}}$ and $\tilde{\sigma}_{\mathbf{j}}$, one easily finds the 2-form $\sigma(x,y)$,

$$\sigma(x,y) = \left(\frac{\partial}{\partial y}\left(\frac{1}{y-x}\frac{\sqrt{P(y)}}{\sqrt{P(x)}}\right) - \frac{\partial}{\partial x}\left(\frac{1}{x-y}\frac{\sqrt{P(x)}}{\sqrt{P(y)}}\right)\right)\mathrm{d}x\,\mathrm{d}y. \tag{C.11}$$

This form is exact, so, apparently the integrals over all 2-cycles must vanish. However, there is a singularity at $x = y$ which produces the intersection number on the right-hand side of (C.10). All this is quite standard, so we do not go into much detail.

Consider a particular case of (C.10),

$$\int_{c_{\mathbf{i}}}\int_{c_{\mathbf{j}}}\sigma(x,y)=0,\qquad \mathbf{i},\mathbf{j}=\mathbf{1},\ldots,\mathbf{n}-\mathbf{1}. \tag{C.12}$$

This is true because the a-cycles do not intersect.

On the product of two copies of Riemann surface we have the canonical second kind differential $\rho(x,y)$ with the following properties.

- The differential $\rho(x,y)$ is holomorphic everywhere except the diagonal, where it has a double pole with no residue

$$\rho(x,y)=\left(\frac{1}{(x-y)^2}+O(1)\right)\mathrm{d}x\,\mathrm{d}y. \tag{C.13}$$

- The differential $\rho(x,y)$ is normalized with respect to x,

$$\int_{c_{\mathbf{m}}}\rho(x,y)=0,\qquad \mathbf{m}=\mathbf{1},\ldots,\mathbf{n}-\mathbf{1}. \tag{C.14}$$

An important consequence of the Riemann bilinear relations is that this differential is automatically symmetric:

$$\rho(x,y)=\rho(y,x). \tag{C.15}$$

Let us explain this by giving an explicit construction of $\rho(x,y)$. We start with an exact form in x,

$$-\frac{\partial}{\partial x}\left(\frac{\sqrt{P(x)}}{\sqrt{P(y)}(x-y)}\right)\mathrm{d}x\,\mathrm{d}y.$$

which obviously has the required singularity at $x=y$, but has also additional singularities at infinity. Because of (C.9) and (C.11), these singularities are cancelled in the following expression:

$$\rho(x,y)=-\frac{\partial}{\partial x}\left(\frac{\sqrt{P(x)}}{\sqrt{P(y)}(x-y)}\right)\mathrm{d}x\,\mathrm{d}y+\sum_{\mathbf{i}=1}^{\mathbf{n}-1}\tilde{\sigma}_{\mathbf{i}}(x)\sigma_{\mathbf{i}}(y)+\sum_{\mathbf{i},\mathbf{j}=\mathbf{1}}^{\mathbf{n}-1}X_{\mathbf{i},\mathbf{j}}\sigma_{\mathbf{j}}(x)\sigma_{\mathbf{i}}(y),$$

where the matrix $X_{\mathbf{i},\mathbf{j}}$ must be defined from the normalization condition

$$\sum_{\mathbf{j}=1}^{\mathbf{n}-1}X_{\mathbf{i},\mathbf{j}}\int_{c_{\mathbf{k}}}\sigma_{\mathbf{j}}+\int_{c_{\mathbf{k}}}\tilde{\sigma}_{\mathbf{i}}=0.$$

Now writing a similar formula for $\rho(y,x)$, it becomes apparent that symmetry (C.15) is equivalent to the fact that X is a symmetric matrix. This fact follows from (C.12). There is an obvious similarity between this argument and the proof of lemma 7.2.

Suppose that we want to construct a normalized second kind differential with given singular part. To be more precise, we allow a singularity only at $x=1$ with a given singular part

$$\tau_{\mathrm{sing}}(x)=\sum_{k=2}^{N}\gamma_k(x-1)^{-k}\,\mathrm{d}x.$$

So, we look for a differential which has the singular part $\tau_{\mathrm{sing}}(x)$ at $x=1$ and is holomorphic elsewhere. We require that $\tau(x)$ is normalized

$$\int_{c_{\mathbf{m}}}\tau(x)=0.$$

It is rather obvious that $\tau(x)$ is given by

$$\tau(x) = \oint_{\Gamma} \sigma(x, y)\, \mathrm{d}^{-1}\tau_{\mathrm{sing}}(y), \tag{C.16}$$

where the contour Γ is as usual: 1 is inside it and x outside.

Let us return to the quasi-classical limit of the quantum formulae. First, note that for $\alpha = 0$ the operator $\overline{D}$ becomes the second difference derivative because $\rho(\zeta) = 1$:

$$\frac{1}{(\pi \mathrm{i}\nu)^2}\overline{D}_{\zeta}(f(\zeta)) = \frac{1}{(\pi \mathrm{i}\nu)^2}(f(\zeta q) + f(\zeta q^{-1}) - 2f(\zeta)) \underset{\nu\to 0}{\to} \left(\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^2 f(\zeta).$$

Also Δ_{ζ}^{-1} goes to the primitive function

$$2\pi \mathrm{i}\nu\Delta_{\zeta}^{-1}(f(\zeta)) \underset{\nu\to 0}{\to} \left(\zeta\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^{-1} f(\zeta).$$

Consider the $f(\zeta) = L(\zeta^2)$ and the corresponding q-deformed exact form (for $\alpha = 0$ there is no difference between $f^{\pm}(\zeta)$):

$$\varpi_{\nu}(\zeta^2) = \frac{1}{\pi \mathrm{i}\nu}E(f(\zeta))Q^{-}(\zeta)Q^{+}(\zeta)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}$$

then

$$\varpi_{\nu}(z) \underset{\nu\to 0}{\to} -\frac{\mathrm{d}}{\mathrm{d}z}(L(z)\sqrt{P(z)})\,\mathrm{d}z.$$

Denote

$$\sigma_{\nu}(\zeta^2, \xi^2) = \frac{1}{\pi \mathrm{i}\nu}r(\zeta, \xi)Q^{-}(\zeta)Q^{+}(\zeta)\varphi(\zeta)Q^{-}(\xi)Q^{+}(\xi)\varphi(\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2}\frac{\mathrm{d}\xi^2}{\xi^2},$$

Then we have

$$\sigma_{\nu}(x, y) \underset{\nu\to 0}{\to} \sigma(x, y) + \tfrac{1}{2}(\sigma_{\mathbf{0}}(x)\tilde{\sigma}_{\mathbf{0}}(y) - \sigma_{\mathbf{0}}(y)\tilde{\sigma}_{\mathbf{0}}(x)), \tag{C.17}$$

the additional term is not important in (C.10) because $\tilde{\sigma}_{\mathbf{0}}$ is an exact form. The limit (C.17) explains the name q-deformed Riemann bilinear relations for (5.8).

Consider now

$$\rho_{\nu}(\zeta^2, \xi^2) = \frac{1}{\pi \mathrm{i}\nu}T(\zeta)T(\xi)\omega(\zeta, \xi)Q^{-}(\zeta)Q^{+}(\zeta)\varphi(\zeta)Q^{-}(\xi)Q^{+}(\xi)\varphi(\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2}\frac{\mathrm{d}\xi^2}{\xi^2}.$$

We want to show that

$$\rho_{\nu}(x, y) \underset{\nu\to 0}{\to} \rho(x, y).$$

First, it is rather easy to find that in singularity (6.5) two simple poles produce in the classical limit the double pole in (C.13). Second, we have the normalization conditions (6.10). They look different from the normalization conditions (C.14) because of the presence of the term

$$\frac{1}{\pi \mathrm{i}\nu}\int_{\Gamma_{\mathbf{m}}} T(\zeta, \kappa)\overline{D}_{\zeta}\overline{D}_{\xi}\Delta_{\zeta}^{-1}\psi(\zeta/\xi)Q^{-}(\zeta)Q^{+}(\zeta)\varphi(\zeta)\frac{\mathrm{d}\zeta^2}{\zeta^2}. \tag{C.18}$$

However, this term for $\alpha = 0, \nu \to 0$ is of order ν^2, while $\rho_{\nu}(\zeta^2, \xi^2)$ is of order 1. So term (C.18) does not count and from (6.10) with $\mathbf{m} = \mathbf{1}, \ldots, \mathbf{n} - \mathbf{1}$ we get the normalization conditions (C.14). Conditions (6.10) with $\mathbf{m} = \mathbf{0}, \mathbf{n}$ show that the differential $\rho_{\nu}(\zeta^2, \xi^2)$ in the limit $\nu \to 0$ does not have simple poles at $\zeta^2 = 0, \infty$ which were originally present.

Thus we conclude that the function $\omega(\zeta, \xi)$ is related in the classical limit to the canonical normalized second kind differential.

Note a clear similarity between formula (C.16) and our main formula (6.1).

Appendix D. Equivalence of different non-degeneracy conditions

In this appendix, we show that the conditions $\det(\mathcal{A}^{\pm}) \neq 0$ are equivalent to the fact that the scalar product (2.3) does not vanish. We use usual notations of the quantum inverse scattering method (QISM) [15]:

$$T_{a,\mathbf{M}}(\zeta) = \begin{pmatrix} A(\zeta) & B(\zeta) \\ C(\zeta) & D(\zeta) \end{pmatrix}_a .$$

Consider the case when all the spaces in the Matsubara direction are two-dimensional (spin $1/2$). The basis of the two-dimensional space will be denoted by $e_{\pm}$. Introduce two vectors in Matsubara space

$$|+\rangle = e_+ \otimes \cdots \otimes e_+, , \qquad |-\rangle = e_- \otimes \cdots \otimes e_-. \tag{D.1}$$

The eigenvector $|\kappa\rangle$ is written in QISM framework as

$$|\kappa\rangle = \prod C\big(\lambda_{\mathbf{j}}^-\big)|-\rangle, \tag{D.2}$$

where $\big(\lambda_{\mathbf{j}}^-\big)^2$ are zeros of $\zeta^\kappa Q_{\mathbf{M}}^-(\zeta,\kappa)$ which is a polynomial of ζ^2. It is well known that this eigenvector does not vanish identically unless $\tau_{\mathbf{i}} = \tau_{\mathbf{j}} q$ for some $\mathbf{j} > \mathbf{i}$. The latter situation has to be forbidden from the very beginning because the tensor product on **i**th and **j**th spaces is reducible and contains one-dimensional sub-module. On the other hand, there is no problem with the case $\tau_{\mathbf{i}} = \tau_{\mathbf{j}} q^{-1}$ which allows the fusion procedure, and show that our considering only spin $1/2$ representations is not a real restriction.

Consider now the vector $\prod B\big(\lambda_{\mathbf{j}}^+\big)|+\rangle$, where $\big(\lambda_{\mathbf{j}}^+\big)^2$ are zeros of $\zeta^{-\kappa} Q_{\mathbf{M}}^+(\zeta,\kappa)$. This vector also does not vanish identically, it is an eigenvector of $T_{\mathbf{M}}(\zeta,\kappa)$ with the same eigenvalue as (D.2). Hence, the assumed uniqueness of the eigenvector with the eigenvalue of maximal absolute value implies that this vector is proportional to $|\kappa\rangle$ with some coefficient which depends on τ_j and κ, the exact form of this coefficient is irrelevant here.

Now consider the scalar product (2.3). We do not care about the normalization of the eigenvectors, so, in traditional QISM way it is written as

$$\langle \kappa + \alpha|\kappa\rangle = \langle -| \prod B\big(\mu_{\mathbf{j}}^-\big) \prod C\big(\lambda_{\mathbf{j}}^+\big)|-\rangle,$$

where $\big(\mu_{\mathbf{j}}^{\pm}\big)^2$ are zeros of $\zeta^{\mp\kappa} Q_{\mathbf{M}}^{\pm}(\zeta,\kappa)$. Due to the previous remark we rewrite

$$\langle \kappa + \alpha|\kappa\rangle = \mathrm{const}\langle -| \prod B\big(\mu_{\mathbf{j}}^-\big) \prod B\big(\lambda_{\mathbf{j}}^+\big)|+\rangle, \tag{D.3}$$

where const is a non-vanishing constant which was discussed above. So, we conclude that the scalar product in question is given essentially by the partition function with domain wall boundary conditions

$$M_{\mathbf{n}}(\xi_{\mathbf{1}},\ldots,\xi_{\mathbf{n}}|\tau_{\mathbf{1}},\ldots,\tau_{\mathbf{n}}) = \prod \xi_{\mathbf{j}}^{-1}\langle -| \prod_{\mathbf{j}=1}^{\mathbf{n}} B(\xi_{\mathbf{j}})|+\rangle$$

with specification $\{\xi_{\mathbf{j}}\} = \{\mu_{\mathbf{j}}^-\} \cup \big\{\lambda_{\mathbf{j}}^+\big\}$, note that independently of spin of our eigenvectors the number of elements in the latter set is $\mathbf{n}$.

Being a polynomial of degree $\mathbf{n}-\mathbf{1}$ in $\xi_{\mathbf{n}}^2$ the function $M_{\mathbf{n}}$ is completely characterized by the recurrence relation:

$$\begin{aligned} M_{\mathbf{n}}(\xi_{\mathbf{1}},\ldots,\xi_{\mathbf{n-1}},\tau_{\mathbf{n}}|\tau_{\mathbf{1}},\ldots\tau_{\mathbf{n-1}},\tau_{\mathbf{n}}) &= (q^2-1)\tau_{\mathbf{n}}^{-1} \prod \tau_{\mathbf{j}}^{-2} \\ &\times \prod_{\mathbf{j}\neq\mathbf{n}} \big(q^2\xi_{\mathbf{j}}^2 - \tau_{\mathbf{n}}^2\big)\big(q^2\tau_{\mathbf{n}}^2 - \tau_{\mathbf{j}}^2\big) M_{\mathbf{n-1}}(\xi_{\mathbf{1}},\ldots,\xi_{\mathbf{n-1}}|\tau_{\mathbf{1}},\ldots,\tau_{\mathbf{n-1}}). \end{aligned} \tag{D.4}$$

This recurrence was solved by Izergin who found a determinant formula for $M_{\mathbf{n-1}}$ [16].

On the other hand, we have the determinant $\det(\mathcal{A}^{+})$ of $(\mathbf{n}+\mathbf{1})\times(\mathbf{n}+\mathbf{1})$ matrix. This determinant depends on the Bethe roots only through the product $Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)$. Once again we consider the union $\{\xi_{\mathbf{j}}\}=\{\mu_{\mathbf{j}}^{-}\}\cup\{\lambda_{\mathbf{j}}^{+}\}$ and normalize this product as follows:

$$Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)=\prod_{\mathbf{j}=1}^{\mathbf{n}}\left(\zeta^{2}-\xi_{\mathbf{j}}^{2}\right).$$

The determinant can be reduced in two steps:

$$\begin{aligned}\det\left(\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}=\mathbf{0},\ldots,\mathbf{n}}&=-2\pi\mathrm{i}\prod\xi_{\mathbf{j}}^{2}\det\left(\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}=\mathbf{1},\ldots,\mathbf{n}},\\ \det\left(\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}=\mathbf{1},\ldots,\mathbf{n}}&=-2\pi\mathrm{i}\det\left(\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}=\mathbf{1},\ldots,\mathbf{n}-\mathbf{1}},\end{aligned}\tag{D.5}$$

where we used the obvious identities:

$$\int_{\Gamma_{\mathbf{0}}}\zeta^{\alpha+2\mathbf{j}}Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^{2}}{\zeta^{2}}=(-1)^{\mathbf{n}-1}2\pi\mathrm{i}\delta_{\mathbf{j},\mathbf{0}}\prod\xi_{\mathbf{j}}^{2},$$

$$\int_{\Gamma_{\infty}}\zeta^{\alpha+2\mathbf{j}}Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)\varphi(\zeta)\frac{\mathrm{d}\zeta^{2}}{\zeta^{2}}=-2\pi\mathrm{i}\delta_{\mathbf{j},\mathbf{n}},$$

Making the dependence on $\mathbf{n}$ and other parameters explicit we introduce

$$\begin{aligned}&D_{\mathbf{n}}(\xi_{1},\ldots,\xi_{\mathbf{n}}|\tau_{1},\ldots,\tau_{\mathbf{n}})\\ &\quad=(-1)^{\mathbf{n}(\mathbf{n}-1)/2}\prod\tau_{\mathbf{j}}^{-2}\prod_{\mathbf{i},\mathbf{j}}\left(q\tau_{\mathbf{i}}^{2}-q^{-1}\tau_{\mathbf{j}}^{2}\right)\prod_{\mathbf{i}<\mathbf{j}}\left(\tau_{\mathbf{i}}^{2}-\tau_{\mathbf{j}}^{2}\right)\det\left(\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}}\right)_{\mathbf{i},\mathbf{j}=\mathbf{1},\ldots,\mathbf{n}}.\end{aligned}$$

where we preferred the intermediate reduction from (D.5) for its antisymmetry with respect to permutation of τs. In the case of two-dimensional representations in the Matsubara direction the integrals in $\mathcal{A}^{+}_{\mathbf{i},\mathbf{j}}$ are easy: they are given by the sum of two residues. Obviously, $D_{\mathbf{n}}$ is a polynomial in $\xi_{\mathbf{n}}^{2}$ of degree $\mathbf{n}$. However the second relation from (D.5) shows that the actual degree is $\mathbf{n}-\mathbf{1}$.

Set $\xi_{\mathbf{n}}=\tau_{\mathbf{n}}$ and multiply the matrix $\mathcal{A}^{+}$ from the right by the matrix $I-\tau_{\mathbf{n}}^{2}E$ with $E_{i,j}=\delta_{i,j-1}$. Then it is easy to see that in the last row only $\mathbf{n}$th matrix element does not vanish. Using this, after some simple algebra one sees that $D_{\mathbf{n}}$ satisfies relation (D.4). Hence we conclude that

$$D_{\mathbf{n}}(\xi_{1},\ldots,\xi_{\mathbf{n}}|\tau_{1},\ldots,\tau_{\mathbf{n}})=M_{\mathbf{n}}(\xi_{1},\ldots,\xi_{\mathbf{n}}|\tau_{1},\ldots,\tau_{\mathbf{n}}).$$

Due to the above reasoning it shows that $\langle\kappa+\alpha|\kappa\rangle$ is proportional to $\det(\mathcal{A}^{+})$ with a non-vanishing coefficient. Similarly, rewriting $\langle\kappa+\alpha|\kappa\rangle$ as

$$\langle\kappa+\alpha|\kappa\rangle=\mathrm{const}\langle+|\prod C\left(\mu_{\mathbf{j}}^{+}\right)\prod C\left(\lambda_{\mathbf{j}}^{+}\right)|-\rangle,$$

one proves that it is proportional to $\det(\mathcal{A}^{-})$ with a non-vanishing coefficient.

References

[1] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2007 Hidden Grassmann structure in the XXZ model *Commun. Math. Phys.* **272** 263–81

[2] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2008 Hidden Grassmann structure in the XXZ model II: creation operators *Commun. Math. Phys.* at press

[3] Boos H, Jimbo M, Miwa T, Smirnov F and Takeyama Y 2007 Fermionic basis for space of operators in the XXZ model *SISSA Proc. Sci.* Paper 015 (34 pp, electronic)

[4] Boos H, Göhmann F, Klümper A and Suzuki J 2007 Factorization of the finite temperature correlation functions of the XXZ chain in a magnetic field *J. Phys. A: Math. Theor.* **40** 10699–727

[5] Bazhanov V, Lukyanov S and Zamolodchikov A 1999 Integrable structure of conformal field theory: III. The Yang–Baxter relation *Commun. Math. Phys.* **200** 297–324

[6] Klümper A 1992 Free energy and correlation length of quantum chains related to restricted solid-on-solid lattice models *Ann. Phys., Lpz.* **1** 540–53

[7] Kitanine N, Maillet J-M and Terras V 2000 Correlation functions of the XXZ Heisenberg spin-$\frac{1}{2}$-chain in a magnetic field *Nucl. Phys.* B **567** 554–82

[8] Göhmann F, Klümper A and Seel A 2004 Integral representations for correlation functions of the XXZ chain at finite temperature *J. Phys. A: Math. Gen.* **37** 7625–52

[9] Suzuki M 1985 Transfer matrix method and Monte Carlo simulation in quantum spin systems *Phys. Rev.* B **31** 2957–65

[10] Smirnov F 1998 Structure of matrix elements in quantum Toda chain *J. Phys. A: Math. Gen.* **31** 8953–71

[11] Smirnov F 2004 Baxter equations and deformation of Abelian integrals *Int. J. Mod. Phys.* A **19** (Suppl.) 396–417

[12] Smirnov F 2000 Quasi-classical study of form facotrs in finite volume *Am. Math. Soc. Trans.* **201** 283–307

[13] Sklyanin E 1992 Quantum inverse scattering method. Selected topics *Quantum Groups and Quantum Integrable Systems (Nankai Lectures in Mathematical Physics)* ed M-L Ge (Singapore: World Scientific)

[14] Mukhin E, Tarasov V and Varchenko A 2007 Bethe algebra of homogeneous XXX Heisenberg model has simple spectrum arXiv:0706.0688

[15] Sklyanin E, Takhtajan L and Faddeev L 1979 Quantum inverse problem method *Theor. Math. Phys.* **40** 688–706

[16] Izergin A 1987 Partition function of a six-vertex model in a finite volume *Sov. Phys. Dokl.* **32** 878–9